Measure and Integration: Mid-Term, 2020-21

(1) Let X be a set and $\mathcal{F}$ a collection of real valued functions on X satisfying the following properties:
 (i) $\mathcal{F}$ contains the constant functions,
 (ii) if $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in \mathcal{F}$,
 (ii) if $f_n \in \mathcal{F}$, and $f = \lim\limits_{n\to\infty} f_n$, then $f \in \mathcal{F}$.

For $A \subseteq X$, denote by $\mathbf{1}_A$ the indicator function of A, i.e.

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ \\ 0, & x \notin A. \end{cases}$$

Show that the collection $\mathcal{A} = \{A \subseteq X : \mathbf{1}_A \in \mathcal{F}\}$ is a σ-algebra.

(2) Let X be a set. We call collection $\mathcal{F}$ of subsets of X an algebra if the following conditions hold: (i) $\emptyset \in \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and (iii) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

 (a) Let $\mathcal{F}_1 \subset \mathcal{F}_1 \subset \cdots$ be a strictly increasing sequence of algebras on X. Show that $\displaystyle\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is an algebra on X.

 (b) Let μ be a **pre**-measure on $\displaystyle\bigcup_{n=1}^{\infty} \mathcal{F}_n$. Find a measure ν on $\sigma\Big(\displaystyle\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big)$ extending μ, i.e. $\mu(A) = \nu(A)$ for all $A \in \displaystyle\bigcup_{n=1}^{\infty} \mathcal{F}_n$.

(3) Let $(X, \mathcal{D}, \mu)$ be a measure space, and let $\overline{\mathcal{D}}^{\mu}$ be the completion of the σ-algebra $\mathcal{D}$ with respect to the measure μ (see exercise 4.15). We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{D}}^{\mu}$. Suppose $f : X \to X$ is a function such that $f^{-1}(B) \in \mathcal{D}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{D}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}^{\mu}$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{D}}^{\mu}$.

(4) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{3k+2^n}{2^n} \cdot \mathbf{1}_{[k/2^n,(k+1)/2^n)}(x),\ n \geq 1.$$

Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in \mathbb{R}$.