Measure and Integration: Practice Final Exam 2020-21

(1) Consider the measure space $[1,\infty),\mathcal{B}([1,\infty]),\lambda)$ where $\mathcal{B}([1,\infty])$ is the Borel σ-algebra and λ is the Lebesgue measure restricted to $[1,\infty)$. Show that

$$\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)=1.$$

(Hint: $\lim\limits_{x\to 0}\sin(x)/x=1$)

(2) Let $(X,\mathcal{A},\mu)$ be a measure space, and $p,q\in(1,\infty)$ and $r\geq 1$ be such that $1/r=1/p+1/q$. Show that if $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $fg\in\mathcal{L}^r(\mu)$ and $||fg||_r\leq||f||_p\,||g||_q$.

(3) Consider the function $u:(1,2)\times\mathbb{R}\to\mathbb{R}$ given by $u(t,x)=e^{-tx^2}\cos x$. Let λ denotes Lebesgue measure on $\mathbb{R}$, show that the function $F:(1,2)\to\mathbb{R}$ given by $F(t)=\int_{\mathbb{R}}e^{-tx^2}\cos x\,d\lambda(x)$ is differentiable.

(4) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Let $k,g\in\mathcal{L}^1(\lambda)$ and define $F:\mathbb{R}^2\to\mathbb{R}$, and $h:\mathbb{R}\to\overline{\mathbb{R}}$ by

$$F(x,y)=k(x-y)g(y).$$

(a) Show that F is measurable.
(b) Show that $F\in\mathcal{L}^1(\lambda\times\lambda)$, and

$$\int_{\mathbb{R}\times\mathbb{R}}F(x,y)d(\lambda\times\lambda)(x,y)=\left(\int_{\mathbb{R}}k(x)d\lambda(x)\right)\left(\int_{\mathbb{R}}g(y)d\lambda(y)\right).$$

(5) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f\in\mathcal{L}^1(\lambda)$ and define for $h>0$, the function $f_h(x)=\dfrac{1}{h}\displaystyle\int_{[x,x+h]}f(t)\,d\lambda(t)$.

(a) Show that f_h is Borel measurable for all $h>0$.
(b) Show that $f_h\in\mathcal{L}^1(\lambda)$ and $||f_h||_1\leq||f||_1$.

(6) Let $(X,\mathcal{A},\mu)$ be a measure space, and $p\in[1,\infty)$. Let $f,f_n\in\mathcal{L}^p(\mu)$ satisfy $\lim\limits_{n\to\infty}||f_n-f||_p=0$, and $g,g_n\in\mathcal{M}(\mathcal{A})$ satisfy $\lim\limits_{n\to\infty}g_n=g$ μ a.e. Assume that $|g_n|\leq M$, where $M>0$ is a real number. Show that $\lim\limits_{n\to\infty}||f_ng_n-fg||_p=0$.