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Measure and Integration: Mid-Term, 2024-25

(1) Let $(X, \mathcal{A}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$) and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$ measurable map.

(a) Let $\mathcal{C} = \{A \in \mathcal{A} : A = T^{-1}A\}$. Prove that $\mathcal{C}$ is a σ-algebra. (1 pt)

(b) Let $\mathcal{D} = \{A \in \mathcal{A} : \mu(A) = \mu(T^{-1}A)\}$. Prove that $\mathcal{D}$ is a Dynkin system, and that $\mathcal{C} \subseteq \sigma(\mathcal{D}) \subseteq \mathcal{A}$. (1.5 pts)

(c) Suppose $\{\nu_n\}$ is an increasing sequence of measures defined on $\mathcal{A}$ (here increasing means that $\nu_n(A) \leq \nu_{n+1}(A)$ for all $A \in \mathcal{A}$ and all $n \in \mathbb{N}$). Define ν on $\mathcal{A}$ by $\nu(A) = \lim_{n\to\infty} \nu_n(A)$. Prove that ν is a measure. (1.5 pts)

(2) Let $(X, \mathcal{A}, \mu)$ be a measure space. For $A \in \mathcal{A}$ let

$$S(A) = \{B \in \mathcal{A} : B \subseteq A, \ \mu(B) < \infty\}.$$

Define $\nu : \mathcal{A} \to [0, \infty]$ by $\nu(A) = \sup\{\mu(B) : B \in S(A)\}$.

(a) Show that ν is monotone, i.e. if $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subseteq A_2$, then $\nu(A_1) \leq \nu(A_2)$. (0.5 pts)

(b) Show that if $A \in \mathcal{A}$ with $\mu(A) < \infty$, then $\nu(A) = \mu(A)$. (1 pt)

(c) Show that ν is σ-subadditive on $\mathcal{A}$. (1.5 pts)

(3) Consider the measure space $([0, 1), \mathcal{B}([0, 1)), \lambda)$, where $\mathcal{B}([0, 1))$ is the Borel σ-algebra restricted to $[0, 1)$ and λ is the restriction of Lebesgue measure on $[0, 1)$. Define a function $F : [0, 1) \to [0, 1)$ by

$$F(x) = \sum_{n=0}^{\infty} \left(\frac{3^{n+1}x}{2} - \frac{1}{2}\right) \cdot \mathbb{1}_{[3^{-(n+1)}, 3^{-n})}(x),$$

where $\mathbb{1}_A$ denotes the indicator function of the set A.

(a) Show that F is $\mathcal{B}([0, 1))/\mathcal{B}([0, 1))$ measurable. (1 pt)

(b) Prove that the image measure $F(\lambda) = \lambda \circ F^{-1}$ satisfies $F(\lambda) = \lambda$. (2 pts)