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Measure and Integration: Final, 2023-24

You are allowed to bring the old three A4's (two-sided) used for the mid-term and a new three A4's (two-sided) covering the material after the mid-term

(1) Let $(X, \mathcal{A}, \mu)$ be a probability space and $u \in \mathcal{M}(\mathcal{A})$. Assume that $u^2, u^4 \in \mathcal{L}^1(\mu)$ with $\int u^4\, d\mu = \int u^2\, d\mu = 1$.

(a) Prove that $(u^2-1)^2 \in \mathcal{L}^1(\mu)$ and that $\int (u^2-1)^2\, d\mu = 0$. (1 pt)

(b) Prove that there exists a simple function $f \in \mathcal{E}$ taking three values $-1, 0$ and 1 such that $u = f$ μ a.e. (1.5 pts)

(c) Prove that for $n \geq 1$,

$$\int u^{2n}\, d\mu = 1 \text{ and } \int u^{2n-1}\, d\mu = \mu(\{x \in X : u(x) = 1\}) - \mu(\{x \in X : u(x) = -1\}).$$

(1 pt)

(2) Consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is Lebesgue measure on $[0,1]$. Define $u_n(x) = \dfrac{n^2 x \sin(nx)}{1+n^4x^2}$ for $x \in [0,1]$ and $n \geq 1$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} u_n\, d\lambda(x) = 0.$$

(1.5 pts)

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty)$.

(a) Let $(u_n)_{n\geq 1}$ be a sequence in $\mathcal{L}^\infty(\mu)$ such that $\sum_{n\geq 1} \|u_n\|_\infty < \infty$. Prove that $\sum_{n\geq 1} u_n \in \mathcal{L}^\infty(\mu)$ and that $\|\sum_{n\geq 1} u_n\|_\infty \leq \sum_{n\geq 1} \|u_n\|_\infty$. (1 pt)

(b) Prove that if $u \in \mathcal{L}^p(\mu)$, then $\lim_{n\to\infty} n^p \mu(\{x \in X : |u(x)| > n\}) = 0$. (1.5 pts)

(4) Let $X = Y = \mathbb{Z}_+ = \{0, 1, 2, 3, \cdots\}$ and $(a_n)_{n\geq 0}$ a sequence of positive real numbers such that $\sum_{n\geq 0} a_n < \infty$. Let $\mathcal{A}$ be the collection of all subsets of $\mathbb{Z}_+$ and $\mu_1 = \mu_2$ be counting measure on $\mathbb{Z}_+$, i.e. $\mu_1 = \mu_2 = \sum_{n\geq 0} \delta_n$, where δ_n is Dirac measure concentrated at n. Let $u : X \times Y \to \mathbb{R}$ be defined by

$$u(n,m) = \begin{cases} 1 + a_n & \text{if } n = m, \\ -1 - a_n & \text{if } n = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that $\int_X \int_Y u(n,m)\, d\mu_2(m)\, d\mu_1(n) = 1 + a_0$. (1 pt)

(b) Prove that $\int_Y \int_X u(n,m)\, d\mu_1(n)\, d\mu_2(m) = a_0$. (1 pt)

(c) Explain why parts (a) and (b) do not contradict Fubini's Theorem. (0.5 pt)