Universiteit Utrecht
Mathematisch Instituut

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration: Mid-Term, 2023-24

3 (1) Let X be a set and $\mathcal{C}$ a non-empty collection of subsets of X. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra of X containing $\mathcal{C}$, and let

$$\mathcal{A} = \{A \in \sigma(\mathcal{C}) : A \in \sigma(\mathcal{C}_0) \text{ for some countable collection } \mathcal{C}_0 \subseteq \mathcal{C}\},$$

(the collection $\mathcal{C}_0$ depends on the set A, and by countable we mean empty, finite or infinitely countable).

(a) Show that $\mathcal{A}$ is a σ-algebra over X. (2 pts)

(b) Show that $\mathcal{A} = \sigma(\mathcal{C})$. (1 pt)

3.5 (2) Let $(X, \mathcal{A}, \mu)$ be a measure space. For any $B \in \mathcal{A}$ with $0 < \mu(B) < \infty$, define $\mu_B : \mathcal{A} \to [0, \infty)$ by

$$\mu_B(A) = \frac{\mu(B \cap A)}{\mu(B)}.$$

(a) Prove that μ_B is a measure on $(X, \mathcal{A})$ for any $B \in \mathcal{A}$ with $0 < \mu(B) < \infty$. Conclude that the triple $(X, \mathcal{A}, \mu_B)$ is a probability space. (1 pt)

(b) Assume that $\mu(X) < \infty$ and that $X = \dot{\bigcup}_{n=1}^{\infty} B_n$ (disjoint union) with $B_n \in \mathcal{A}$ and $\mu(B_n) > 0$.

(i) Prove that for any $A \in \mathcal{A}$, one has

$$\mu(A) = \sum_{n=1}^{\infty} \mu_{B_n}(A)\mu(B_n).$$

(1.5 pts)

(ii) Prove that for any for any $i \geq 1$ and any $A \in \mathcal{A}$ with $\mu(A) > 0$, one has

$$\mu_A(B_i) = \frac{\mu_{B_i}(A)\mu(B_i)}{\sum_{n=1}^{\infty} \mu_{B_n}(A)\mu(B_n)}.$$

(1 pt)

3.5 (3) Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Let $\{[a_n, b_n) : n \in \mathbb{N}\}$ be a countable **partition** of $[0,1)$ such that $\lambda([a_n, b_n)) > 0$. Define a function $f : [0,1) \to [0,1)$ by

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x - a_n}{b_n - a_n}\right) \cdot \mathbb{I}_{[a_n, b_n)}(x),$$

where $\mathbb{I}_A$ denotes the indicator function of the set A.

(a) Show that f is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (1 pt)

(b) Prove that $\lambda(f^{-1}([a,b))) = b - a$ for any interval $[a,b)$ in $[0,1)$ with $a < b$. What is the value of $\lambda(f^{-1}([a,b)))$ if $b \leq a$? (1.5 pts)

(c) Prove that the image measure $f(\lambda) = \lambda \circ f^{-1}$ satisfies $f(\lambda) = \lambda$. (1 pt)