Mathematisch Instituut, Faculteit Wiskunde en Informatica, UU.
In electronische vorm beschikbaar gemaakt door de $\mathcal{TBC}$ van A–Es[2].
Het college MAAT werd in 2000/2001 gegeven door Prof. Dr. Richard Gill.

Maat en Integratie (MAAT)
1 december 2000

Tentamen Maat- en Integratietheorie, kerstmis 2000.

1. Consider the real number $x \in [0,1)$ expressed in base-p where $p \in \mathbb{Z}^+$. That is to say

$$x = \sum_{k\geq 1} \frac{\alpha_k}{p^k} \text{ where } \alpha_k \in \{0, ..., p-1\}$$

and can thus be represented in the form $x = [0.\alpha_1\alpha_2\alpha_3...]_p$. Suppose now that $y = [0.\alpha_1\alpha_2\alpha_3...\alpha_n]_p$ is the rational number determined by the first n coefficients in the base p expansion of x. Let $z = y + p^{-n}$ and call $\mathcal{P}_n(x) = [y, z)$ *the base-p cylinder of order n containing* x. Suppose for some $\mathbb{Z}^+ \ni q \neq p$ we define analogously $\mathcal{Q}_m(x)$ to be the base-q cylinder of order m containing x. Now let

$$m_{\mathcal{P},\mathcal{Q}}(n,x) = \sup\{m : \mathcal{P}_n(x) \subseteq \mathcal{Q}_m(x)\}$$

so that $m_{\mathcal{P},\mathcal{Q}}(n,x)$ is the largest order of a base-q cylinder containing the order n base-p cylinder containing x (note that as m increases $\mathcal{Q}_m(x)$ becomes shorter). The object of this exercise is to prove the following

Claim: *Let λ be Lebesgue measure on $[0,1)$, then*

$$\lim_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} = \frac{\log_2 p}{\log_2 q} \quad \lambda\text{-almost everywhere.}$$

a. Show that if

$$\mathbb{Z} \ni m' > n(1+\varepsilon)\frac{\log_2 p}{\log_2 q}$$

then $\lambda(\mathcal{P}_n(x)) > \lambda(\mathcal{Q}_{m'}(x))$ and thus conclude that

$$\limsup_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \leq (1+\varepsilon)\frac{\log_2 p}{\log_2 q} \quad \text{for all } x \in [0,1).$$

b. Now let

$$m^*(n) = \lceil n(1-\varepsilon)\frac{\log_2 p}{\log_2 q}\rceil.$$

Show that for all $x \in [0,1)$

$$\frac{\lambda(\mathcal{P}_n(x))}{\lambda\left(\mathcal{Q}_{m^*(n)}(x)\right)} < \alpha \times 2^{-\zeta n} \quad \text{for some } \zeta > 0$$

where α is some positive constant.

c. Now define

$$\Gamma_n = \{x \in [0,1) : \mathcal{P}_n(x) \nsubseteq \mathcal{Q}_{m^*(n)}(x)\}$$

and show using part b. that $\lambda(\Gamma_n) < 2 \times \alpha \times 2^{-\zeta n}$. [*Note that the intersection of Γ_n with any $\mathcal{Q}_{m^*(n)}(x)$ is at most two non-overlapping subintervals, each subinterval touching an end point of $\mathcal{Q}_{m^*(n)}(x)$*].

d. Use the Borel-Cantelli Lemma to conclude that

$$\liminf_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,x)}{n} \geq (1-\varepsilon)\frac{\log_2 p}{\log_2 q} \quad \text{for } \lambda\text{-almost all } x \in [0,1)$$

and thus together with part a. conclude that the claim is true.

2. The aim of this exercise is to prove the existence of Lebesgue measure on the Borel sets of the unit interval, via Carathéodory's extension theorem. You therefore should not use the existence of this measure or its properties, in establishing the results below.

   Let $\mathcal{A}$ denote the set of finite unions of disjoint intervals of the form $[s,t) \subseteq [0,1)$. Define $\mu(A)$, for $A \in \mathcal{A}$, as the sum of the lengths of the intervals forming A.

a. Show that $\sigma(\mathcal{A}) = \mathcal{B}_{[0,1)}$ and that $\mathcal{B}_{[0,1)} = \{B \cap [0,1) : B \in \mathcal{B}_{\mathbb{R}}\}$.

b. Argue that $\mathcal{A}$ is an algebra but not a σ-algebra, and that μ is a finitely additive, finite measure on $\mathcal{A}$.

c. Suppose that $A_n \in \mathcal{A}$ satisfies $A_n \downarrow \emptyset$ and that $\mu(A_n) \downarrow c > \varepsilon > 0$. Let $\varepsilon_n > 0$ satisfy $\sum \varepsilon_n \leq \varepsilon$. Let k_n be 2 times the number of disjoint intervals making up A_n, and let $\mathcal{E}_n$ denote the set of k_n endpoints of all these intervals. For $x \in [0,1]$, let $d_n(x)$ denote the distance from the point x to the nearest point of $\mathcal{E}_n$. Let $B_n = \{x \in A_n : d_n(x) \geq \varepsilon_n/k_n\}$. Define $g_n(x) = 0$ for $x \notin A_n$, $g_n(x) = 1$ for $x \in B_n$, $g_n(x) = k_n d_n(x)/\varepsilon_n$ for $x \in A_n \setminus B_n$. Argue that g_n is continuous, and that $g_n = 1$ on a union of disjoint intervals of total length at least $c - \varepsilon_n$.

d. Define $f_n = \min_{m \leq n} g_m$. Argue that f_n is continuous on $[0,1]$, that $f_n = 1$ on a union of disjoint intervals of length at least $c - \varepsilon > 0$, and that $f_n \downarrow 0$ for $n \to \infty$.

e. Use Dini's lemma to obtain a contradiction. Use Carathéodory's extension theorem to conclude the existence of Lebesgue measure on $\mathcal{B}_{[0,1)}$.