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# Measure and Integration Mid-term Exam
## Due date: April 19

1. Let $\phi : [A, B] \to [a, b]$ be a strictly increasing surjective continuous function. Suppose $\psi : [a, b] \to \mathbb{R}$ is non-decreasing, and $f : [a, b] \to \mathbb{R}$ a bounded $\psi$-Riemann integrable function. Define $\alpha$ and $g$ on $[A, B]$ by
$$\alpha(y) = \psi(\phi(y)) \text{ and } g(y) = f(\phi(y)).$$
Show that $g$ is $\alpha$-Riemann integrable, and
$$\int_A^B g\, d\alpha = \int_a^b f\, d\psi.$$

2. Let $\{c_n\}$ be a sequence satisfying $c_n \geq 0$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{s_n\}$ be a sequence of distinct points in $(a, b)$. Define a function $\psi$ on $[a, b]$ by $\psi(x) = \sum_{n=1}^{\infty} c_n 1_{(s_n,b]}(x)$, where $1_{(s_n,b]}$ is the indicator function of the interval $(s_n, b]$. Prove that any continuous function $f$ on $[a, b]$ is $\psi$-Riemann integrable, and
$$\int_a^b f(x) d\psi(x) = \sum_{n=1}^{\infty} c_n f(s_n).$$

3. Let $\Gamma \subseteq \mathbb{R}^n$. Recall that the inner Lebesque measure of $\Gamma$ is defined by
$$|\Gamma|_i = \sup\{|K| : K \subseteq \Gamma,\ K \text{ is compact}\}.$$
Prove the following.
   (a) $\Gamma$ is Lebesgue measurable **if and only if** $|\Gamma|_e = |\Gamma|_i$.
   (b) $\Gamma$ is Lebesgue measurable **if and only if** $|A|_e = |\Gamma \cap A|_e + |\Gamma^c \cap A|_e$ for all $A \subseteq \mathbb{R}^n$.
   (c) If $A \subseteq \Gamma$, and $\Gamma$ is Lebesgue measurable, then $|A|_e + |\Gamma \setminus A|_i = |\Gamma|$.

4. Let $E$ be a set, and $\mathcal{A}$ an algebra over $E$. Let $\mu : \mathcal{A} \to [0, 1]$ be a function satisfying
(I) $\mu(E) = 1 = 1 - \mu(\emptyset)$,
(II) if $A_1, A_2, \cdots, \in \mathcal{A}$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then
$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

(a) Show that if $\{A_n\}$ and $\{B_n\}$ are increasing sequences in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$, then $\lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \mu(B_n)$.

(b) Let $\mathcal{G}$ be the collection of all subsets $G$ of $E$ such that there exists an increasing sequence $\{A_n\}$ in $\mathcal{A}$ with $G = \bigcup_{n=1}^{\infty} A_n$. Define $\overline{\mu}$ on $\mathcal{G}$ by

$$\overline{\mu}(G) = \lim_{n\to\infty} \mu(A_n),$$

where $\{A_n\}$ is an increasing sequence in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n$. Show the following.

(i) $\overline{\mu}$ is well defined.

(ii) If $G_1, G_2 \in \mathcal{G}$, then $G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{G}$ and

$$\overline{\mu}(G_1 \cup G_2) + \overline{\mu}(G_1 \cap G_2) = \overline{\mu}(G_1) + \overline{\mu}(G_2).$$

(iii) If $G_n \in \mathcal{G}$ and $G_1 \subseteq G_2 \subseteq \cdots$, then $\bigcup_{n=1}^{\infty} G_n \in \mathcal{G}$ and

$$\overline{\mu}(\bigcup_{n=1}^{\infty} G_n) = \lim_{n\to\infty} \overline{\mu}(G_n).$$

(c) Define $\mu^*$ on $\mathcal{P}(E)$ (the power set of $E$) by

$$\mu^*(A) = \inf\{\overline{\mu}(G) : A \subseteq G,\ G \in \mathcal{G}\}.$$

(i) Show that $\mu^*(G) = \overline{\mu}(G)$ for all $G \in \mathcal{G}$, and

$$\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*(A) + \mu^*(B)$$

for all subsets $A, B$ of $E$. Conclude that $\mu^*(A) + \mu^*(A^c) \geq 1$ for all $A \subseteq E$.

(ii) Show that if $C_1 \subseteq C_2 \subseteq \cdots$ are subsets of $E$ and $C = \bigcup_{n=1}^{\infty} C_n$, then $\mu^*(C) = \lim_{n\to\infty} \mu^*(C_n)$.

(iii) Let $\mathcal{H} = \{B \subseteq E : \mu^*(B) + \mu^*(B^c) = 1\}$. Show that $\mathcal{H}$ is a $\sigma$-algebra over $E$, and $\mu^*$ is a measure on $\mathcal{H}$.

(iv) Show that $\sigma(E; \mathcal{A}) \subseteq \mathcal{H}$. Conclude that the restriction of $\mu^*$ to $\sigma(E; \mathcal{A})$ is a measure extending $\mu$, i.e. $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

5. Let $\overline{\mathcal{B}}_{\mathbb{R}^N}$ be the Lebesgue $\sigma$-algebra over $\mathbb{R}^N$, $\mathcal{B}_{\mathbb{R}^N}$ the Borel $\sigma$-algebra over $\mathbb{R}^N$, and $\mathcal{B}_{\overline{\mathbb{R}}}$ the Borel $\sigma$-algebra over $\overline{\mathbb{R}} = [-\infty, \infty]$. Denote by $\lambda_{\mathbb{R}^N}$ the Lebesgue measure on $\overline{\mathcal{B}}_{\mathbb{R}^N}$. Let $f : \mathbb{R}^N \to [-\infty, \infty]$ be Lebesgue measurable (i.e. $f^{-1}(A) \in \overline{\mathcal{B}}_{\mathbb{R}^N}$ for all $A \in \mathcal{B}_{\overline{\mathbb{R}}}$). Show that there exists a function $g : \mathbb{R}^N \to [-\infty, \infty]$ which is Borel measurable (i.e. $g^{-1}(A) \in \mathcal{B}_{\mathbb{R}^N}$ for all $A \in \mathcal{B}_{\overline{\mathbb{R}}}$) such that

$$\lambda_{\mathbb{R}^N}\left(\{x \in \mathbb{R}^N : f(x) \neq g(x)\}\right) = 0.$$

(Hint: assume first that $f$ is a non-negative simple function)

6. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable **simple** function such that $\int_E f\, d\mu < \infty$. Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\mu(A) < \delta$ then $\int_A f\, d\mu < \epsilon$.