INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UU.
MADE AVAILABLE IN ELECTRONIC FORM BY THE $\mathcal{TBC}$ OF A−Eskwadraat
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Measure and Integration (WISB 312)
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Question 1

Let $f, g : [a, b] \to \mathbb{R}$ be bounded Riemann integrable funcions. Show that fg is Riemann integrable. (Hint: express fg in terms of $(f + g)$ and $(f - g)$).

Question 2

Consider the measure space $(\mathbb{R}, \overline{\mathcal{B}}_{\mathbb{R}}, \lambda)$, where $\overline{\mathcal{B}}_{\mathbb{R}}$ is the Lebesgue σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)},\ n \geq 1.$$

a) Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$.

b) Let $f(x) = \lim_{n\to\infty} f_n(x)$, for $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is measurable.

c) Show that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) d\lambda(x) = \frac{1}{2}$.

Question 3

Let $M \subset \mathbb{R}$ be a non-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}}_{\mathbb{R}}$.). Define $A = \{(x, x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be given by $g(x) = (x, x)$.

a) Show that $A \in \overline{\mathcal{B}}_{\mathbb{R}^2}$. i.e. A is Lebesgue measurable. (Hint: use the fact that Lebesgue measure is rotation invariant).

b) Show that g is a Borel-measurable function, i.e. $g^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for each $B \in \mathcal{B}_{\mathbb{R}^2}$.

c) Show that $A \notin \mathcal{B}_{\mathbb{R}^2}$, i.e. A is not Borel measurable.

Question 4

Let $\mathcal{M} = \{E \subseteq \mathbb{R} : |A|_e = |A \cap E|_e + |A \cap E^c|_e$ for all $A \subseteq \mathbb{R}\}$, where $|A|_e$ denotes the outer Lebesgue measure of A.

a) Show that $\mathcal{M}$ is an algebra over $\mathbb{R}$. (Hint: $A \cap (E_1 \cup E_2) = (A \cap E_1) \bigcup (A \cap E_2 \cap E_1^c)$).

b) Prove by induction that if $E_1, \cdots, E_n \in \mathcal{M}$ are pairwise disjoint, then for any $A \subseteq \mathbb{R}$

$$|A \cap (\bigcup_{i=1}^{n} E_i)|_e = \sum_{i=1}^{n} |A \cap E_i|_e.$$

c) Show that if $E_1, E_2, \cdots \in \mathcal{M}$ is a countable collection of disjoint elements of $\mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

d) Show that $\mathcal{M}$ is a σ-algebra over $\mathbb{R}$.

e) Let $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$. Show that $\mathcal{C} \subseteq \mathcal{M}$. Conclude that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$, where $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ-algebra over $\mathbb{R}$.