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In 2004/2005, the course WISB312 was given by Karma Dajani.

Measure and Integration, re-exam (WISB312)
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Question 1

Let E, F be sets and let $\mathcal{C}$ be a collection of subsets of F. Suppose $T : E \to F$ is a function, and let

$$T^{-1}(\sigma(\mathcal{C})) = \{T^{-1}A : A \in \sigma(\mathcal{C})\}$$

where $\sigma(\mathcal{C})$ is the σ-algebra over F generated by $\mathcal{C}$. Show that $T^{-1}(\sigma(\mathcal{C}))$ is a σ-algebra over E, and that $T^{-1}(\sigma(\mathcal{C})) = \sigma(T^{-1}\mathcal{C})$, where $T^{-1}\mathcal{C} = \{T^{-1}A : A \in \mathcal{C}\}$.

Question 2

Suppose that μ and ν and λ are finite measures on $(E, \mathcal{B})$ such that $\mu \ll \nu$ and $\nu \ll \lambda$. Show that $\mu \ll \lambda$, and that $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}$ λ a.e.

Question 3

Let ν be a σ-finite measure on $(E, \mathcal{B})$, and suppose $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(\Gamma) = \sum_{n=1}^{\infty} 2^{-n}\nu(\Gamma \cap E_n)/(\nu(E_n) + 1)$.

a) Prove that μ is a finite measure on $(E, \mathcal{B})$ which is equivalent to ν.

b) Determine explicitly two measurable functions f and g such that $f = \frac{d\mu}{d\nu}$ and $g = \frac{d\nu}{d\mu}$ ν a.e.

Question 4

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ the Lebesgue measure.

a) Let $f : \mathbb{R} \to \bar{\mathbb{R}}$ be measurable, and suppose $\int_{\mathbb{R}} f(x)d\lambda(x)$ exists. Show that for all $a \in \mathbb{R}$, one has

$$\int_{\mathbb{R}} f(x-a)d\lambda(x) = \int_{\mathbb{R}} f(x)d\lambda(x)$$

b) Let $k, g \in L^1(\lambda)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \bar{\mathbb{R}}$ by

$$F(x,y) = k(x-y)g(y) \text{ and } h(x) = \int_{\mathbb{R}} F(x,y)d\lambda(y)$$

1. Show that F is measurable.
2. Show that $\lambda(|h| = \infty) = 0$ and $\int_{\mathbb{R}} |h(x)|d\lambda(x) \leq \left(\int_{\mathbb{R}} |k(x)|d\lambda(x)\right) \left(\int_{\mathbb{R}} |g(y)|d\lambda(y)\right)$.

Question 5

Consider the measure space $([0,\infty),\mathcal{B},\lambda)$, where $\mathcal{B}$ and λ are the restriction of the Borel σ-algebra and Lebesgue measure to the interval $[0,\infty)$. Define for $n \geq 1$, $f_n : [0,\infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n+\pi & \text{if } n \leq x \leq n + \frac{1}{2n} \\ \pi & \text{otherwise.} \end{cases}$$

a) Prove that $f_n \to \pi$ λ a.e. and in λ-measure.

b) Prove that $\lim_{m\to\infty} \lambda(\sup_{n\geq m} |f_n - \pi| \geq \epsilon) = \infty$ for all $\epsilon > 0$.