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Maat en integratie (WISB312)
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Question 1

Let $\mathbb{Q}$ be a set of all real rational numbers, and let $\mathcal{I}_{\mathbb{Q}} = \{[a,b)_{\mathbb{Q}} : a,b \in \mathbb{Q}\}$ where $[a,b)_{\mathbb{Q}} = \{q \in \mathbb{Q} : a \leq q < b\}$.

a) Prove that $\sigma(\mathcal{I}_{\mathbb{Q}}) = \mathcal{P}(\mathbb{Q})$, where $\mathcal{P}(\mathbb{Q})$ is the collection of all subsets of $\mathbb{Q}$.

b) Let μ be counting measure on $\mathcal{P}(\mathbb{Q})$, and let $\nu = 2\mu$. Show that $\nu(A) = \mu(A)$ for all $A \in \mathcal{I}_{\mathbb{Q}}$, but $\nu \neq \mu$ on $\sigma(\mathcal{I}_{\mathbb{Q}}) = \mathcal{P}(\mathbb{Q})$. Why doesn't this contradict Theorem 5.7 in your book?

Question 2

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $X, \mathcal{A}^*, \bar{\mu})$ be its completion (see exercise 4.13).

a) Let $f \in \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A}^*)$, and $A \in \mathcal{A}^*$ a $\bar{\mu}$-null set (i.e. $\bar{\mu}(A) = 0$). Suppose $g : X \to \bar{\mathbb{R}}$ is a function satisfying $f(x) = g(x)$ for all $x \notin A$. Prove that $g \in \mathcal{M}_{\bar{\mathbb{R}}}(\mathcal{A}^*)$.

b) Let $h \in \mathcal{E}(\mathcal{A}^*)$. Prove that there exists a function $f \in \mathcal{E}(\mathcal{A})$ such that $\{x \in X : h(x) \neq f(x)\} \in \mathcal{A}^*$ and $\bar{\mu}(\{x \in X : h(x) \neq f(x)\}) = 0$.

Question 3

Let X be a set. A collection $\mathcal{A}$ of subsets of X is an algebra if (i) $\emptyset \in \mathcal{A}$, (ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$ and (iii) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. A collection $\mathcal{M}$ of sets is said to be a *M-class* if it satisfies the following two properties:

I. if $\{A_n\} \subseteq \mathcal{M}$ with $A_1 \subseteq A_2 \subseteq \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$, and

II. if $\{B_n\} \subseteq \mathcal{M}$ with $B_1 \supseteq B_2 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} B_n \in \mathcal{M}$.

a) Show that the intersection of an arbitrary collection of M-classes is an M-class.

b) Let X be a set, and $\mathcal{B}$ a collection of subsets of X. Show that $\mathcal{B}$ is a σ-algebra if and only if $\mathcal{B}$ is an algebra and an M-class.

c) Let $\mathcal{A}$ be an algebra over X, and $\mathcal{M}$ the smallest M-class containing $\mathcal{A}$, i.e. $\mathcal{M}$ is the intersection of all M-classes containing $\mathcal{A}$.

   i. Show that $\mathcal{M}_1 = \{B \subset X : B^c \in \mathcal{M}$, and $B \cup A \in \mathcal{M}$ for all $A \in \mathcal{A}\}$ is an M-class containing $\mathcal{A}$. Conclude that $\mathcal{M} \subset \mathcal{M}_1$.

   ii. Show that $\mathcal{M}_2 = \{B \subset X : B^c \in \mathcal{M}$, and $B \cup M \in \mathcal{M}$ for all $M \in \mathcal{M}\}$ is an M-class containing $\mathcal{A}$. Conclude that $\mathcal{M}$ is an algebra.

d) Using the same notation as in part c), show that $\mathcal{M} = \sigma(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the smallest σ-algebra over X containing the algebra $\mathcal{A}$.

Question 4

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathcal{A})$. Consider the measure ν defined on $\mathcal{A}$ by $\nu(A) = \int \mathbf{1}_A u \, d\mu$.