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Measure and Integration (WISB-312)
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Question 1

Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ the Lebesgue measure.

a) Let $f \in \mathcal{L}^1(\lambda)$. Show that for all $a \in \mathbb{R}$, one has

$$\int_{\mathbb{R}} f(x-a)d\lambda(x) = \int_{\mathbb{R}} f(x)d\lambda(x)$$

b) Let $k, g \in \mathcal{L}^1(\lambda)$. Define $F : \mathbb{R}^2 \to \bar{\mathbb{R}}$ by

$$F(x,y) = k(x-y)g(y) \text{ and } h(x) = \int_{\mathbb{R}} F(x,y)d\lambda(y).$$

- Show that F is measurable.
- Show that
$$\int_{\mathbb{R}} |h(x)|d\lambda(x) \leq \left(\int_{\mathbb{R}} |k(x)|d\lambda(x)\right)\left(\int_{\mathbb{R}} |g(y)|d\lambda(y)\right)$$
and $\lambda(|h| = \infty) = 0$.

Question 2

Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ and λ are the restrictions of the Borel σ-algebra and the Lebesgue measure to the interval $(0,\infty)$. Show that

$$\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x} d\lambda(x) = 1.$$

Question 3

Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$).

a) Suppose $1 \leq p < r$, and $f_n, f \in \mathcal{L}^r(\mu)$ satisfy $\lim_{n\to\infty} \|f_n - f\|_r = 0$. Show that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

b) Assume $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $f_n, f \in \mathcal{L}^p(\mu)$, and $g_n, g \in \mathcal{L}^q(\mu)$ satisfy

$$\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \|g_n - g\|_q = 0$$

Show that $\lim\limits_{n\to\infty} \|f_n g_n - fg\|_1 = 0$.

Question 4

Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x,t) = e^{-xt}$ that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes the Lebesgue measure.

Question 5

Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are non-negative. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f \, d\mu_1 \qquad \text{and} \qquad \nu_2(B) = \int_B g \, d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

a) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x, y) \in D\}$. Show that if $\mu_1(D_y) = 0$ ν_1-a.e., then $\mu_2(D_y) = 0$ ν_2-a.e.

b) Show that if $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$ then $(\mu_2 \times \nu_2)(D) = 0$.

c) Show that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y) \, d(\mu_1 \times \nu_1)(x, y).$$