F. Ziltener

blok 1, 2014/ 2015
November 5, 2014

Functionaalanalyse, WISB315

Tentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use results that were proved in the lecture or in the book by Rynne and Youngson, without proving them.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

31 points will yield a passing grade 6, and 61 points a grade 10.

Good luck!

1	2	3	4	5	6	7	8	9	10	11	$\sum$
/4	/4	/6	/5	/3	/4	/7	/3	/3	/7	/15	/61

Problem 1 (polarization identity, 4 pt)**.** Let $(X, \langle\cdot,\cdot\rangle)$ be a Hermitian inner product space. (Hence $\mathbb{K} = \mathbb{C}$.) Show that

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\Big), \quad \forall x, y \in X.$$

Problem 2 (comparing ℓ^p-norms, 4 pt)**.** Let $p \le q \in [1, \infty)$. Show that

$$\ell^p \subseteq \ell^q,$$
$$\|x\|_q \le \|x\|_p, \quad \forall x \in \ell^p.$$

Problem 3 (quotient seminorm, 6 pt)**.** Let $(X, \|\cdot\|)$ be a semi-normed vector space and $Y \subseteq X$ a linear subspace. Prove that the following map is a seminorm:

$$\|\cdot\|^Y : X/Y \to [0, \infty),$$
$$\|x + Y\|^Y := \inf_{y\in Y} \|x - y\|.$$

Remark: You do not need to prove that X/Y is a vector space.

Problem 4 (integration bounded, 5 pt)**.** We equip $C[0, 1]$ with the supremum norm $\|\cdot\|_\infty$. Let $y \in C[0, 1]$. Show that the operator

$$T : C[0, 1] \to \mathbb{K}, \quad Tx := \int_0^1 x(t)y(t)\, dt,$$

is bounded and calculate its operator norm.

Hint: Prove that for a certain constant $C \ge 0$ we have $\|T\| \le C$ and $\|T\| \ge C - \varepsilon$, for every $\varepsilon > 0$.

Problem 5 (equivalence of norms, 3 pt)**.** Let $\|\cdot\|$ and $|||\cdot|||$ be complete norms on a vector space X, such that there exists $C \in \mathbb{R}$ satisfying

$$\|x\| \le C|||x|||, \quad \forall x \in X.$$

Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Hint: Use a result from the lecture.

Problem 6 (dual space of ℓ^∞/c_0, 4 pt)**.** Prove that the dual space of ℓ^∞/c_0, equipped with the quotient norm, is nonzero.

Problem 7 (Hilbert space reflexive, 7 pt)**.** Prove that every Hilbert space H is reflexive.

Hint: Relate the canonical map $\iota_H : H \to H''$ to the map

$$\Phi_H : H \to H', \quad \Phi_H(y) := \langle\cdot, y\rangle.$$

Problem 8 (bidual operator, canonical map, 3pt). Let X, Y be normed spaces and $T \in B(X,Y)$. Prove that

$$T'' \circ \iota_X = \iota_Y \circ T.$$

Problem 9 (adjoint of left-shift, 3 pt). Find the adjoint operator for

$$L : \ell^2 \to \ell^2, \quad Lx := \big(x^2, x^3, \dots\big).$$

Remark: You may use that L is linear and bounded. You may also use the characterization of the adjoint operator that was proved in an assignment.

Problem 10 (spectrum, 7 pt). Let

$$T : X := C\big([0,1], \mathbb{C}\big) \to X, \quad (Tx)(t) := \int_0^t x(s)ds.$$

Show that the spectrum of T equals $\{0\}$.

Hint: Compute the spectral radius of T.

Problem 11 (continuous functions and ℓ^2, 15 pt). Let $A \subseteq [0,1]$ be a closed subset different from $[0,1]$. We define

$$X := \big\{x \in C[0,1] \,\big|\, x(t) = 0, \, \forall t \in A\big\}, \quad \|\cdot\| : X \to \mathbb{R}, \, \|x\| := \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Prove that there exists a linear isometry $T : X \to \ell^2$, whose image is dense.

Remark: In this exercise you may use any exercise from the assignments (and any result from the lecture and the book by Rynne and Youngson).