F. Ziltener

blok 2, 2015/ 2016
January 27, 2016

Functionaalanalyse, WISB315

Tentamen

Family name: ____________________ Given name: ____________________

Student number: ______________________________________

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof that

- a given map is linear (if this is indeed the case),
- a given normed space is separable/ inseparable.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

32 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	9	$\sum$
/4	/15	/7	/7	/6	/6	/6	/6	/7	/64

Problem 1 (ℓ^1 and inner product, 4 pt). Does there exist an inner product on ℓ^1 that induces the norm $\|\cdot\|_1$?

Problem 2 ($\|\cdot\|_\infty$ complete norm, 15 pt). Show that the map

$$\|\cdot\|_\infty : \ell^\infty \to [0,\infty), \quad \|(x_i)_{i\in\mathbb{N}}\|_\infty := \sup_{i\in\mathbb{N}} |x_i|,$$

is a complete norm.

Problem 3 ($L^2\big((0,1),\mathbb{C}\big)$ and $\ell^2_{\mathbb{C}}$, 7 pt). Prove that there exists a unitary map from $L^2\big((0,1),\mathbb{C}\big)$ to $\ell^2_{\mathbb{C}}$ (=space of square integrable sequences of complex numbers).

Problem 4 (spectrum closed, 7 pt). Let X be a complex Banach space and $T \in B(X)$. Prove that the spectrum $\sigma(T)$ is closed.

Remark: This is a direct consequence of a lemma that was stated in the lecture and proved as an exercise. You need to reprove the statement here.

Problem 5 (canonical map, 6 pt). Show that for every normed vector space X the canonical map $\iota_X : X \to X''$ is an isometry.

Remark: This was part of a proposition in the lecture, whose proof relied on a corollary. You need to reprove the relevant parts of the proposition and the corollary here. You do *not* need to prove that ι_X is well-defined, nor that it is linear.

Problem 6 (criterion for boundedness, 6 pt). Let H be a Hilbert space and $T : H \to H$ a linear map satisfying

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad \forall x, y \in H.$$

Prove that T is bounded.

Hint: Use a result from the lecture.

Problem 7 (c_0 not reflexive, 6 pt). Prove that

$$\Big(c_0 := \big\{(x_i)_{i\in\mathbb{N}} \text{ sequence in } \mathbb{R} \text{ converging to } 0\big\}, \|\cdot\|_\infty\Big)$$

is not reflexive.

Problem 8 (spectrum of left-shift, 6 pt). Find the spectrum of the left-shift operator

$$L : \ell^2 \to \ell^2, \quad Lx := \big(x_2, x_3, \dots\big).$$

Problem 9 (commutator of operators, 7 pt). Do there exist a normed vector space X and bounded linear maps $S, T : X \to X$ such that

$$ST - TS = \mathrm{id}?$$

Remark: You will get partial credit for solving this problem in the easier case $\dim X < \infty$.