F. Ziltener

blok 2, 2015/ 2016
March 23, 2016

Functionaalanalyse, WISB315

Hertentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof that a given map is well-defined or linear (if this is indeed the case).

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

If you are in doubt whether you may use a certain result then please ask!

You may write in Dutch.

22 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	$\sum$
/4	/6	/7	/5	/7	/4	/11	/15	/59

Problem 1 (inner product, 4 pt)**.** Show that the map

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad \langle x, y \rangle := \sum_{i=1}^{n} x^i \overline{y}^i,$$

is an inner product.

Remark: Here you may *not* assume that a given map is linear.

Problem 2 (space of bounded linear maps, 6 pt)**.** Let X, Y be normed spaces. We denote by $B(X, Y)$ the set of bounded linear maps from X to Y.

(i) Show that $B(X, Y)$ is a linear subspace of $\mathrm{Lin}(X, Y)$.

(ii) Show that the map

$$\| \cdot \|_X^Y : B(X, Y) \to \mathbb{R}, \quad \|T\|_X^Y := \sup \left\{ \|Tx\|_Y \,\middle|\, x \in \overline{B}_1^X(0) \right\},$$

is a norm.

Problem 3 (Hilbert space reflexive, 7 pt)**.** Prove that every Hilbert space H is reflexive.

Hint: Relate the canonical map $\iota_H : H \to H''$ to the map

$$\Phi_H : H \to H', \quad \Phi_H(y) := \langle \cdot, y \rangle.$$

Problem 4 (Closed Graph Theorem, 5 pt)**.** Let X, Y be Banach spaces and $T : X \to Y$ a linear map, such that the graph of T is closed (with respect to the product topology). Show that T is bounded.

Remark: You may use the fact that the map

$$\| \cdot \|_1 : X \times Y \to [0, \infty), \quad \|(x, y)\|_1 := \|x\|_X + \|y\|_Y,$$

is a complete norm.

Hint: Use a theorem from the lecture.

Problem 5 (integral operator, 7 pt)**.** We equip $X := C\big([0, 1], \mathbb{C}\big)$ with the supremum norm $\| \cdot \|_\infty$, and define the operator

$$T : X \to X, \quad (Tx)(t) := \int_0^1 e^{st} x(s) ds.$$

Show that T is compact.

Problem 6 (dualization, 4 pt)**.** Let X and Y be normed spaces. Show that the dualization map

$$B(X,Y) \ni T \mapsto T' \in B(Y',X')$$

is an isometry.

Problem 7 (orthonormal basis, 11 pt)**.** Let $\big(X, \langle\cdot,\cdot\rangle\big)$ be an inner product space.

(i) Let $(v_i)_{i\in\mathbb{N}}$ be a sequence in X. Prove that there exists a nondecreasing function $n:\mathbb{N}\to\mathbb{N}_0$ and an orthonormal system $(w_j)_{1\leq j\in n(\mathbb{N})}$ in X, such that

$$\operatorname{span}\big\{w_j \,\big|\, 1\leq j\leq n(k)\big\} = \operatorname{span}\big\{v_i \,\big|\, 1\leq i\leq k\big\}, \quad \forall k\in\mathbb{N}.$$

(ii) Prove that every separable Hilbert space has a countable (maybe finite) orthonormal basis.

Hint for (i): Gram-Schmidt procedure. (You need to prove that this procedure has the desired properties.)

Problem 8 (integral equation, 15 pt)**.** Prove:

(i) There exists a solution $0\not\equiv x\in X := C\big([0,1],\mathbb{C}\big)$ of the equation

$$\int_0^1 e^{st}x(s)ds = 0, \quad \forall t\in[0,1],$$

or there exists $y\in X$ for which there is no solution $x\in X$ of

$$\int_0^1 e^{st}x(s)ds = y(t), \quad \forall t\in[0,1].$$

(ii) There exist a countable (or finite) subset $S\subseteq[-e,e]$ ($e=$ Euler's number) and for each $\lambda\in S$ a function $0\not\equiv x_\lambda\in X$ with the following properties:

(a)

$$\int_0^1 e^{st}x_\lambda(s)ds = \lambda x_\lambda(t), \quad \forall t\in[0,1].$$

(b) For every $\lambda\in\mathbb{C}\setminus(S\cup\{0\})$ and $y\in X$ there exists a unique solution $x\in X$ of the equation

$$\int_0^1 e^{st}x(s)ds - \lambda x(t) = y(t), \quad \forall t\in[0,1].$$

Remark: In this problem you may use any exercise from the assignments, any other problem of this exam, and any result from the lecture and the book by Rynne and Youngson.