F. Ziltener

blok 2, 2016/ 2017
April 21, 2017

Functionaalanalyse, WISB315

Hertentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use the following without proof:

- A given map is linear (if this is indeed the case).
- For every $p \in [1, \infty]$ the normed space $(\ell^p, \|\cdot\|_p)$ is complete.
- For all $1 \leq p \leq q \leq \infty$, $x \in \ell^p$ we have
$$\ell^p \subseteq \ell^q, \quad \|x\|_q \leq \|x\|_p.$$
- Every subset of a separable metric space is separable.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

25 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	$\sum$
/9	/4	/5	/5	/13	/4	/8	/13	/61

Problem 1 (vector space, 9 pt)**.** Let S be a set. We denote by $V := \mathbb{R}^S$ the set of functions from S to $\mathbb{R}$. We define the maps

$$p : V \times V \to V, \quad p(v,w)(s) := v(s) + w(s),$$
$$m : \mathbb{R} \times V \to V, \quad m(a,v)(s) := a \cdot (v(s)).$$

Show that (V, p, m) is an $\mathbb{R}$-vector space.

Remark: p and m are pointwise addition and scalar multiplication. If you prefer then you may use the more customary notation $v + w := p(v,w)$ and $av := a \cdot v := m(a,v)$. Then the above formulae become $(v + w)(s) := v(s) + w(s)$, $(a \cdot v)(s) := a \cdot (v(s))$.

Problem 2 (compactness, 4 pt)**.** Is the closed unit ball in $\ell^1 = \ell^1(\mathbb{N})$ compact?

Problem 3 (pointwise limit of continuous operators, 5 pt)**.** Let X be a Banach space, Y a normed space, and $T_n : X \to Y$ be a bounded linear operator, for $n \in \mathbb{N}$. Assume that for every $x \in X$ the limit

$$T(x) := \lim_{n\to\infty} T_n(x)$$

exists. Show that the operator T is bounded. (It is linear, but you do not need to show this.)

Problem 4 (nonopen map, 5 pt)**.** Find a Banach space $(X, \|\cdot\|_X)$, a normed space $(Y\|\cdot\|_Y)$, and a surjective bounded linear map $T : X \to Y$ that is not open.

Remark: Recall that T is called open iff it maps every open set to an open set.

Problem 5 (dual operator, 13 pt)**.** Let X, Y, Z be normed spaces, $T \in B(X,Y)$, and $S \in B(Y,Z)$. Prove:

(i)
$$(S \circ T)' = T' \circ S'.$$

(ii) If T is an isomorphism then T' is an isomorphism and
$$(T')^{-1} = (T^{-1})'.$$

(iii) If T is an isometric isomorphism then the same holds for T'.

Remark: Here we denote by $B(X,Y)$ the set of bounded linear operators from X to Y, and by T' the dual operator of T.

Hints: Use (i) to solve (ii) and (ii) to solve (iii). (You may do this even if you could not solve (i) or (ii).)

Problem 6 (bidual operator, canonical map, 4 pt)**.** Let X, Y be normed spaces and $T \in B(X,Y)$. Prove that

$$T'' \circ \iota_X = \iota_Y \circ T.$$

Here $\iota_X : X \to X''$ denotes the canonical map.

(More problems on the back.)

Problem 7 (dual space of c_0, 8 pt)**.** Show that the map

$$\Phi_{c_0} : \ell^1 \to c_0', \quad \Phi_{c_0}(y)(x) := \sum_{i=1}^{\infty} x^i y^i,$$

is well-defined and isometric.

Remarks: c_0' denotes the dual space of c_0. You do **not** need to show that Φ_{c_0} is linear.

Problem 8 (spectrum, 13 pt)**.** Let $\emptyset \neq K \subseteq \mathbb{C}$ be a compact subset and $p \in [1, \infty]$. Find an operator $T \in B(\ell^p, \ell^p)$ with

$$\sigma(T) = K.$$

(Recall that $\ell^p = \ell^p(\mathbb{N})$.)