Functional Analyse ‖ a

Problem 1 (norm on space of functions, 5 pt). Let $k \in \mathbb{N} \cup \{0\}$. We define

$$X := \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ is } k \text{ times continuously differentiable, } f^{(i)} \text{ is bounded, } \forall i = 0, \dots, k\},$$
$$\|\cdot\| : X \to [0,\infty), \quad \|f\| := \max_{i=0,\dots,k} \sup_{x\in\mathbb{R}} |f^{(i)}(x)|,$$

where $f^{(i)}$ denotes the i-th derivative of f. ($f^{(0)} = f$) Show the following:

(i) X is a linear subspace of the space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(ii) The map $\|\cdot\|$ is a norm.

Problem 2 (orthogonal complement, 4 pt). Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. We define the *orthogonal complement* of A to be the set

$$A^\perp := \{x \in X \,|\, \langle x, y\rangle = 0, \forall y \in A\}.$$

Show that $A^\perp$ is a linear subspace of X and that it is closed in X.

Problem 3 (multiplication operator, 6 pt). Let $p \in [1,\infty)$, and $x \in \ell^\infty = \ell^\infty(\mathbb{N})$. We define

$$M_x : \ell^p \to \ell^p, \quad M_x(y) := xy.$$

Show the following:

(i) The map M_x is well-defined, i.e., $M_x(y) \in \ell^p$ for every $y \in \ell^p$.

(ii) M_x is linear.

(iii) The operator norm of M_x is given by

$$\|M_x\| = \|x\|_\infty.$$

Problem 4 (ℓ^p separable, 5 pt). Prove that for every $p \in [1,\infty)$ the space $\ell^p = \ell^p(\mathbb{N})$ is separable.

Problem 5 (prescribed Fourier coefficients, 4 pt). We call a function $f : [0,1] \to \mathbb{R}$ Lebesgue-measurable iff the set $f^{-1}((-\infty, b])$ is Lebesgue-measurable for every $b \in \mathbb{R}$. Does there exist a Lebesgue-measurable function $f : [0,1] \to \mathbb{R}$, such that

$$\int_{[0,1]} |f|^2 d\lambda < \infty, \quad \widehat{f}^n = \frac{1}{\sqrt[4]{n}}, \forall n \in \mathbb{N}?$$

Remark: Here λ denotes the Lebesgue-measure on $[0,1]$ and $\widehat{f}^n$ the n-th Fourier coefficient of f.

(More problems on the back.)

Problem 6 (completeness of the quotient seminorm, 9 pt). Let X be a real vector space and $Y \subseteq X$ a linear subspace. We denote by X/Y the quotient of X by Y. Let $\|\cdot\|$ be a seminorm on X. We define the quotient seminorm to be the map

$$\|\cdot\|^Y : X/Y \to [0, \infty), \quad \|\overline{x}\|^Y := \inf_{x \in \overline{x}} \|x\|.$$

Show that if $\|\cdot\|$ is complete then $\|\cdot\|^Y$ is complete.

Remark: You do *not* need to show that $\|\cdot\|^Y$ is indeed a seminorm.

Problem 7 (Hilbert space reflexive, 7 pt). Prove that every Hilbert space H is reflexive.

Hint: Relate the canonical map $\iota_H : H \to H''$ to the map

$$\Phi_H : H \to H', \quad \Phi_H(y) := \langle \cdot, y \rangle.$$

Problem 8 (dual space of ℓ^∞, 5 pt). Prove that the map

$$\Phi_\infty : \ell^1 \to (\ell^\infty)', \quad (\Phi_\infty y)x := \sum_{i=1}^{\infty} x^i y^i,$$

$\left(1, \frac{1}{2}, \frac{1}{4}, \dots\right)$

is not surjective.

Remark: You do *not* need to show that this map is well-defined.

Hint: Use another problem of this exam.

Problem 9 (spectrum of integral operator, 7 pt). Let

$$T : X := C([0,1], \mathbb{C}) \to X, \quad (Tx)(t) := \int_0^t x(s) ds.$$

Show that the spectrum of T equals $\{0\}$.

Remark: The map T is well-defined and linear. You do *not* need to show this.

Hint: Compute the spectral radius of T.

Problem 10 (continuous bilinear map, 6 pt). Let $(X_i, \|\cdot\|_i)$, $i = 0, 1$, and $(Y, \|\cdot\|)$ be normed real vector spaces, and $b : X_0 \times X_1 \to Y$ a continuous bilinear map. Assume that X_1 is complete. Show that

$$\sup\left\{ \|b(x_0, x_1)\| \,\middle|\, x_i \in \overline{B}_1^{X_i}(0), \ \forall i = 0, 1 \right\} < \infty.$$