Utrecht University 2023/24

Functional Analysis (WISB315)

Exam 1

Please solve each exercise on a separate sheet of paper and write your name on each. You can use one book of your choice. No other documents nor electronic devices allowed. Veel succes!

Exercise 1. Let X be the space $C[0,1]$ of complex-valued continuous functions on $[0,1]$, equipped with the norm $\|\cdot\|_\infty$ given by

$$\|f\|_\infty = \sup\{|f(t)| \,:\, t \in [0,1]\}, \quad f \in C[0,1].$$

Let Y be the same vector space, equipped with the $\|\cdot\|_2$ norm given by

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}}, \quad f \in C[0,1].$$

Let $I : X \to Y$ be the identity map.

(i) [0,5 p.] Show that $I : X \to Y$ is bijective. Show that $I \in B(X,Y)$.
(ii) [0,5 p.] Find the norm of I in $B(X,Y)$.
(iii) [0,5 p.] For $n \in \mathbb{N}$, let $f_n(t) = t^n$. Compute $\|f_n\|_\infty$ and $\|f_n\|_2$.
(iv) [0,5 p.] Show that the map $I^{-1} : Y \to X$ is not bounded.
(v) [1 p.] Using the previous questions, show that Y is not complete.

Exercise 2. Let X and Y be normed vector spaces (of dimension at least 1).

(i) [1 p.] Justify that there exists $f \in X'$ such that $\|f\| = 1$ and $f(x_0) = 1$ for some $x_0 \in X$ with $\|x_0\| = 1$..
(ii) [0,5 p.] Let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a sequence. For $n \in \mathbb{N}$, let $A_n : X \to Y$ be the linear map defined by

$$A_n x = f(x) y_n, \quad \forall x \in X,$$

where f is as in (i). Show that $A_n \in B(X,Y)$.
(iii) [0,5 p.] Suppose in addition that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Show that $(A_n)_{n\in\mathbb{N}}$ is then a Cauchy sequence in $B(X,Y)$.
(iv) [1 p.] Using the previous questions, show that if $B(X,Y)$ is complete, then Y is complete.

Exercise 3. Let $\mathcal{H}$ be a Hilbert space, and let $T \in B(\mathcal{H})$ be such that $\|T\| \leq 1$.

(i) [0,5 p.] Suppose that $x \neq 0$ is such that $Tx = x$. Show that

$$\|T^*x - x\|^2 = \|T^*x\|^2 - \|x\|^2.$$

(ii) [0,5 p.] Let $x \in \mathcal{H}$ be as in (i). Using (i) and the hypothesis $\|T\| \leq 1$, Show that

$$\|T^*x - x\|^2 \leq 0.$$

Deduce that $T^*x = x$.
(iii) [0,5 p.] Using (ii), show that

$$\mathrm{Ker}(T - I) = \mathrm{Ker}(T^* - I).$$

(iv) [0,5 p.] Show that $(\overline{\mathrm{Im}(T - I)})^\perp = \mathrm{Ker}(T - I)$

Let now $S \in B(\ell^2)$ be given by

$$S(x_1, x_2, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right), \quad (x_n)_{n\in\mathbb{N}} \in \ell^2.$$

(v) [0,5 p.] Show that $\|S\| \leq 1$.
(vi) [1 p.] Show that $0 \in \sigma(S^*)$. *(This part is independent from* (i)–(iv)*)*
(vii) [0.5 p.] Show that $\mathrm{Im}(S - I)$ is dense. *(You can use* (iii)–(iv)*, but this is not mandatory.)*