Solutions for first midterm

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In the solutions, I do not give all possible approaches, but just one or two. Please write to me if you find any typos or have other remarks.

Problem 1 (4 points)**.** *Write* $\frac{1153}{140}$ *as a continued fraction.*

We need to do the Euclidean algorithm:

$$\begin{aligned} 1153 &= 8 \cdot 140 + 33 \\ 140 &= 4 \cdot 33 + 8 \\ 33 &= 4 \cdot 8 + 1 \\ 8 &= 8 \cdot 8 + 0. \end{aligned}$$

Thus, the continued fraction expansion is $[8, 4, 4, 8]$.

Problem 2 (8 points)**.** *Determine for* $n = 1236$ *and* $n = 1153$ *whether they are the sum of two squares. If yes, find one pair of integers* (x, y) *with* $n = x^2 + y^2$*. (Hint: You may use that* $140^2 + 1$ *is divisible by* 1153*.)*

We know that each number is congruent to its digit sum modulo 9. Thus,

$$1236 \equiv 12 \equiv 3 \mod 9.$$

In particular, we see that the exponent of 3 in the prime factorizaion of 1236 is 1, thus odd. Thus, 1236 is not the sum of two squares. [Alternative: The squares modulo 9 are 0, 1, 4 and 7. No sum of two of these is 3.]

In contrast, 1153 is $33^2 + 8^2 = 1089 + 64$. We used Cornacchia's algorithm to find these numbers.

Problem 3 (8 points)**.** *Show that for every natural number* $a > 2$*, there is a Pythagorean triple* (a, b, c)*. (Hint: Distinguish the cases* a *even and* a *odd.)*

The problem is not correct as stated. Indeed if $a \equiv 2 \mod 4$, there is no Pythoagorean triple (a, b, c). Indeed: We have seen in class that if (a, b, c) is a Pythogorean triple, then $a = 2st$ or $a = s^2 - t^2$ for relatively prime s and t that are not both odd. For such s and t, clearly $2st$ is divisible by 4 as s or t is even and thus a cannot be written as $2st$. Moreover, $s^2 - t^2$ is odd and thus a cannot be written as $s^2 - t^2$.

The converse of the quoted theorem also holds: If s and t are relatively prime natural numbers, not both odd, then $(s^2 - t^2, 2st, s^2 + t^2)$ forms a Pythagorean triple. With the help of this theorem, we can solve the cases that a is odd or a divisible by 4.

Suppose that a is odd. We can write it as $2s + 1$ with $s \in \mathbb{N}$. Setting $t = s + 1$, we observe that $a = s^2 - t^2$. Set $b = 2st$ and $c = s^2 + t^2$. Clearly s and t cannot be both odd and are relative prime.

If a is divisible by 4, then we can write it as $4s = 2st$ with $t = 1$ and s even. Again, s and t are relatively prime and not both odd.

If $a \equiv 2 \mod 4$, we still obtain in the same manner b and c such that $a^2 + b^2 = c^2$, but a, b and c are all even and thus not relatively prime.

Problem 4 (8 points)**.** *Let n be a positive integer of the form $8k + 7$ with $k \in \mathbb{Z}$. Show that n is of the form $x^2 + y^2 + z^2 + w^2$ with x, y, z and w **positive** integers.*

First of all we know by Lagrange's four-square theorem that we can find (not necessarily positive) integers x, y, z and w with $n = x^2 + y^2 + z^2 + w^2$. As $(-x)^2 = x^2$, we can assume that $x, y, z, w \geq 0$.

Let $a \in \mathbb{Z}$. We have seen in class that a^2 is congruent to 0, 1 or 4 modulo 8. As $n \equiv 7 \mod 8$, this implies that x^2, y^2, z^2 and w^2 must be (up to reordering) congruent to $1, 1, 1$ and 4 modulo 8, respectively. Thus, x, y, z and w are nonzero and thus positive.