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Solutions[1] Foundations of Mathematics (WISB323) november 5th 2009

Question 1.

a) We verify the defining properties of a filter. Since $x_0 \notin \emptyset$ it follows that $\emptyset \notin \mathcal{F}$. Since $x_0 \in X$ it follows that $X \in \mathcal{F}$. Consider $Y, Z \subseteq X$ and assume $Y \subseteq Z$ and $Y \in \mathcal{F}$. Then $x_0 \in Y$ and thus also $x_0 \in Z$ which implies that $Z \in \mathcal{F}$. Assume now that $Y, Z \in \mathcal{F}$. Then $x_0 \in Y$ and $x_0 \in Z$ thus $x_0 \in Y \cap Z$ which means $Y \cap Z \in \mathcal{F}$. This proves that $\mathcal{F}$ is a filter. Now let $X' \subseteq X$ be arbitrary. Then either $x_0 \in X'$ or $x_0 \notin X'$. In the first case holds $X' \in \mathcal{F}$ and in the second holds $X - X' \in \mathcal{F}$, thus $\mathcal{F}$ is an ultra filter.

b) We apply Zorn's Lemma. Let P be the set of all filters on X containing $\mathcal{F}_0$. Introduce a poset structure on A by inclusions. That is for two filters $\mathcal{F}_1, \mathcal{F}_2 \in P$ define $\mathcal{F}_1 \leq \mathcal{F}_2$ exactly when $\mathcal{F}_1 \subseteq \mathcal{F}_2$. That this is indeed a poset follows trivially from the fact that set inclusion is a reflexive, anti-symmetric, and transitive relation. We need to verify that P is not empty and that it satisfies the chain condition. That P is not empty is clear since evidently $\mathcal{F}_0 \in P$. Now let $\{\mathcal{F}_i\}_{i \in I}$ be a chain in P. Let $\mathcal{F} = \cup_i \mathcal{F}_i$. If $\mathcal{F}$ is in P then it is clearly an upper bound of the chain. Clearly $\mathcal{F}_0 \subseteq \mathcal{F}$ since $\mathcal{F}_0 \subseteq \mathcal{F}_i$ for all $i \in I$. Thus we only need to verify that $\mathcal{F}$ is a filter. Indeed, if $\emptyset \in \mathcal{F}$ then $\emptyset \in \mathcal{F}_i$ for some $i \in I$, but this is not the case since $\mathcal{F}_i$ is a filter. Certainly $X \in \mathcal{F}$ since $X \in \mathcal{F}_i$ for all $i \in I$. Let now $Y, Z \subseteq X$ and assume $Y \subseteq Z$ and $Y \in \mathcal{F}$. Then $Y \in \mathcal{F}_i$ for some $i \in I$ and since $\mathcal{F}_i$ is a filter also $Z \in \mathcal{F}_i$. Thus $Z \in \mathcal{F}$. Now assume that $Y, Z \in \mathcal{F}$. Thus $Y \in \mathcal{F}_i$ and $Z \in \mathcal{F}_j$ for some $i, j \in I$. Since $\{\mathcal{F}_i\}_{i \in I}$ is a chain we can assume without loss of generality that $Y, Z \in \mathcal{F}_i$. But then $Y \cap Z \in \mathcal{F}_i$ and thus $Y \cap Z \in \mathcal{F}$. Thus $\mathcal{F}$ is a filter, as needed. We conclude thus that there exists a maximal element in P. Such a maximal element, by definition, is a filter $\mathcal{F}$ containing $\mathcal{F}_0$ which is maximal with respect to inclusion, as required.

c) Since $\mathbb{N} - \emptyset$ is not finite it follows that $\emptyset \notin \mathcal{F}_{cof}$. Since $\mathbb{N} - \mathbb{N} = \emptyset$ is finite it follows that $\mathbb{N} \in \mathcal{F}_{cof}$. Let $A, B \in \mathcal{F}_{cof}$. Then

$$\mathbb{N} - (A \cap B) = \mathbb{N} \cap (A \cap B)^c = \mathbb{N} \cap (A^c \cup B^c) = (\mathbb{N} \cap A^c) \cup (\mathbb{N} \cap B^c) = (\mathbb{N} - A) \cup (\mathbb{N} - B)$$

and since both $\mathbb{N} - A$ and $\mathbb{N} - B$ are finite it follows that $\mathbb{N} - (A \cap B)$ is finite, thus $A \cap B \in \mathcal{F}_{cof}$. Now assume $B \in \mathcal{F}_{cof}$ and $B \subseteq A$. Then

$$\mathbb{N} - A \subseteq \mathbb{N} - B$$

and since $\mathbb{N} - B$ is finite so is $\mathbb{N} - A$ thus $A \in \mathcal{F}_{cof}$. Thus $\mathcal{F}_{cof}$ is a filter. It is not a principal filter. To verify that it is sufficient to notice that for every $n \in \mathbb{N}$ there exists a set $A_n \in \mathcal{F}_{cof}$ such that $n \notin A_n$. One choice for such an A_n is $A_n = \mathbb{N} - \{n\}$.

d) Consider the filter $\mathcal{F}_{cof}$ on $\mathbb{N}$. By what we have proved there exists a maximal filter $\mathcal{F}$ containing $\mathcal{F}_{cof}$. This filter is non-principal since it is of course still true that for every $n \in \mathbb{N}$ there is a set $A_n \in \mathcal{F}$ with $n \notin A_n$.

Question 2.

a) Consider $\psi_t = \forall x \forall y \forall z (((x \leq y) \wedge (y \leq z)) \rightarrow (x \leq z))$ (which holds if, and only if, $\leq^M$ is transitive), $\psi_{as} = \forall x \forall y (((x \leq y) \wedge (y \leq x)) \rightarrow (x = y))$ (which holds if, and only if, $\leq^M$ is anti-symmetric), and $\psi_r = \forall x (x \leq x)$ (which holds if, and only if, $\leq^M$ is reflexive). Thus the desired theory is $T_{pos} = \{\psi_t, \psi_{as}, \psi_r\}$.

[1] These solutions were made with great precaution. In case of errors, the $\mathcal{T}_B\mathcal{C}$ cannot be held responsible. However, she will be glad to be informed: tbc@a-eskwadraat.nl

b) Let $k \in \mathbb{N}$. Consider the formula $\psi_k(a_1, \cdots, a_n)$ which is equal to the conjunction of all expressions of the form $\neg(a_i \leq a_j)$ with $1 \leq i, j \leq k$ and $i \neq j$. What ψ_k expresses about the variables $a_1, \cdots, a_k$ is that they form an anti-chain. Consider also the formula $\rho_k(b, a_1, \cdots, a_k) = (b = a_1) \vee (b = a_2) \vee \cdots \vee (b = a_k)$ which simply expresses that b is equal to one of the given a_i. One possibility for the desired sentence is now: $\varphi_k = \forall x_1 \forall x_2 \cdots \forall x_k \forall x_{k+1}(\psi_{k+1}(x_1, \cdots, x_{k+1}) \rightarrow \rho(x_1, \cdots, x_{k+1}))$. If there is a anti-chain with more than k elements then for $k+1$ different elements $x_1, \cdots, x_k, x_{k+1}$ from such an anti-chain holds $\psi_{k+1}(x_1, \cdots, x_k, x_{k+1})$ but not $\rho_{k+1}(x_1, x_2, \cdots, x_{k+1})$. As desired.

c) Consider the theory $S = T_{pos} \cup \{\neg\varphi_k \mid k \geq 1\}$ where ψ_k is as in 2). Any model M of S is a poset (since $T_{pos} \subseteq S$) and its width can't be finite since if it were finite, say equal to k, then $M \models \varphi_k$ would hold while $M \models \neg\varphi_m$ for all $m \in \mathbb{N}$. Conversly, any poset M with infinite width arises as a model of S since for such a poset holds $M \models T_{pos}$ (since M is a poset) and for all $m \in \mathbb{N}$ holds $M \models \neg\varphi_k$ since M has anti-chains of arbitrary length.

d) Such a theory T' does not exist. To prove that assume to the contrary that T' exists. Consider the theory $T'' = T' \cup \{\neg\varphi_k \mid k \geq 1\}$. T'' is clearly inconsistent (since a model of it is a poset with finite width (since $T' \subseteq T''$), say equal to k, while at the same time holds $\neg\varphi_k$, namely the poset contains anti-chains larger than k. Clearly a contradiction. By the Compactness Theorem there exists a finite inconsistent theory $S' \subseteq T''$. S' is thus contained in $S'' = T' \cup \{\neg\varphi_k \mid 1 \leq k \leq n\}$ for some $n \in \mathbb{N}$. However, we can easily find models for S''. Indeed, consider a set with $n+1$ elements $U = \{u_0, u_1, \cdots, u_n\}$ with the reflexive relation I as the poset structure. Then U has finite width equal to $n+1$ thus is a model of S''. But then, since $S' \subseteq S''$ it follows that S' is also consistent which is a contradiction since S' is inconsistent.

Question 3.

a) We have to make sure that the definition does not depend on the choice of set X. Thus, let X and Y be sets with $|X| = |Y|$. We have to verify that $|X_{bij}| = |Y_{bij}|$. Since $|X| = |Y|$ there is a bijective function $f : X \rightarrow Y$, thus f has an inverse $f^{-1} : Y \rightarrow X$. We now construct a function $g : X_{bij} \rightarrow Y_{bij}$ and prove that it is bijective, thus proving that $|X_{bij}| = |Y_{bij}|$. For a given $\rho \in X_{bij}$, that is a bijective function $\rho : X \rightarrow X$, let $g(\rho)$ be the function $f \circ \rho \circ f^{-1}$. Then $g(\rho)$ is a function from Y to Y which is bijective since it is a composition of bijective functions. Thus $g(\rho) \in Y_{bij}$. To construct the inverse of g consider $h : Y_{bij} \rightarrow X_{bij}$ given by, for each $\tau \in Y_{bij}$, by $h(\tau) = f^{-1} \circ \tau \circ f$. As before $h(\tau) \in X_{bij}$. Now, for any $\rho \in X_{bij}$ holds

$$h(g(\rho)) = h(f \circ \rho \circ f^{-1}) = f^{-1} \circ f \circ \rho \circ f^{-1} \circ f = \rho$$

and similarly for each $\tau \in Y_{bij}$ holds $g(h(\tau)) = \tau$. Thus $h = g^{-1}$ as required.

b) Let $f : \mathbb{N} \rightarrow \mathbb{N}$. We define $\psi(f) : \mathbb{N} \rightarrow \mathbb{N}$ for each $n \in \mathbb{N}$ as follows:

$$\psi(f)(n) = f(0) + f(1) + \cdots + f(n) + n.$$

We need to prove that $\psi(f)$ is injective and then to prove that ψ is injective. To show that $\psi(f)$ is injective assume to the contrary it is not. Thus there are natural numbers $m, n \in \mathbb{N}$ such that $n < m$ and $\psi(f)(n) = \psi(f)(m)$. But then we have that $f(0) + \cdots + f(n) + n = f(0) + \cdots + f(m) + m$ which implies (since $n < m$) that $0 = f(n+1) + \cdots + f(m) + (m - n)$. But all summands are non-negative and $m - n > 0$ thus the equality is impossible. Thus $\psi(f)$ is indeed injective for all functions $f : \mathbb{N} \rightarrow \mathbb{N}$. Now assume $f_1, f_2 \in \mathbb{N}^{\mathbb{N}}$ are different functions and let n_0 be the smallest natural number for which $f_1(n_0) \neq f_2(n_0)$, without loss of generality assume $f_1(n_0) < f_2(n_0)$. Then

$$\psi(f_1)(n_0) - \psi(f_2)(n_0) = f_1(n_0) - f_2(n_0) < 0$$

thus $\psi(f_1)$ and $\psi(f_2)$ are different functions, so ψ is injective.

c) From the injective map ψ and the given injective map φ we deduce, by definition, that $|\mathbb{N}^{\mathbb{N}}| \leq |\mathbb{N}_{inj}|$ and $|\mathbb{N}_{inj}| \leq |\mathbb{N}_{bij}|$. Since $\mathbb{N}_{inj} \subseteq \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}_{bij} \subseteq \mathbb{N}_{inj}$ the reverse inequalities hold and thus, by the Cantor-Shroeder-Berenstein Theorem, we conclude that $|\mathbb{N}^{\mathbb{N}}| = |\mathbb{N}_{inj}|$ and $|\mathbb{N}_{inj}| = |\mathbb{N}_{bij}|$. Now since the definition of $\omega!$ is independant of the choice of set we can calculate:

$$\omega! = |\mathbb{N}_{bij}| = |\mathbb{N}_{inj}| = |\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|} = 2^{\omega}.$$

The equality preceeding the last one follows from the general property that for any infinite set A holds $|A|^{|A|} = 2^{|A|}$ as is proven in the book (this depends on the axiom of choice).

Question 4.

a) This statement is correct. To prove it we use Zermello's Well Ordering Theorem that states that any set can be well ordered (incidentally, this theorem is equivalent to the axiom of choice). Consider the set $A = \mathbb{R}_+ - \{0\}$. According to Zermello's Theorem there exist a poset structure $\leq_2$ on A such that A is well ordered. Now define an order $\leq_1$ on $\mathbb{R}_+$ as follows: for $x, y \in \mathbb{R}_+$ with $x, y > 0$ define $x \leq_1 y$ if, and only if, $x \leq_2 y$. And for any $z \in \mathbb{R}_+$ define $0 \leq_1 z$. It is trivial that $\leq_1$ well orders $\mathbb{R}_+$ and clearly 0 is the smallest element in that order.

b) This statement is not correct. For a counter example consider the set $\mathbb{N}$ with the usual order, which is well-ordered (this is incidentally equivalent to the principal of induction). Consider the element $0 \in \mathbb{N}$. Then $0_{\uparrow} = \{1, 2, \cdots, n, \cdots\}$ which is clearly order isomorphic to $\mathbb{N}$ (by the function $\mathbb{N} \to 0_{\uparrow}$ which sends n to $n+1$) . Of course, there was nothing special about the number 0 and any other number would have worked equally well.

c) This statement is incorrect. We give two counter examples. First consider the set $\{0\}$ with just one element and the trivial well order structure on it. Then 0 is a limit element and the greatest element. For a slightly less trivial counter example consider the set $\mathbb{N}$ as a well order as in 2) and a new symbol ∞. Define on $\mathbb{N} \cup \{\infty\}$ a poset structure which agrees with the usual order on $\mathbb{N}$ and in which ∞ is a largest element. But then ∞ is a limit element since any element of the form $n+1$ for $n \in \mathbb{N}$ is again a natural number thus different from ∞.