FOUNDATIONS OF MATHEMATICS - EXAM B (21-01-2010)

Voor de Nederlandse tekst van dit tentamen zie ommezijde.

- The use of lecture notes or books is not allowed. The use of one A4 sheet is allowed.

Problem A (Application of the Loewenheim-Skolem Theorem)

(1) Let L be a language with finitely many symbols. Given are two models M_1, M_2 of an L-theory T and an L-sentence φ such that $M_1 \models \varphi$ and $M_2 \models \neg\varphi$. Prove that if both M_1 and M_2 are infinite then there exist two non-isomorphic T-models N_1 and N_2 such that $|N_1| = |N_2|$.

(2) Prove that the condition that M_1 and M_2 are infinite is essential to guarantee the conclusion holds.

Problem B To prove that $\vdash \varphi \leftrightarrow \neg\neg\varphi$ and $\vdash (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$ one needs to construct four proof trees. Of these four choose three and construct them.

Problem C Let ZF be the L-theory of Zermelo-Fraenkel set theory (here $L = \{\in\}$). In parts 1-3 you are asked to give a formula in L that expresses a certain property about its arguments. You do not need to prove your formula is correct. Make sure your formulas are proper L-formulas. In the formation of a formula you may use other formulas from earlier parts. In this question you may use facts about sets proved in part one of the course without giving an axiomatic proof.

(1) $func(x, y, z)$ expresses that x is a function from y to z.

(2) $inj(x, y, z)$ expresses that x is an injective function from y to z.

(3) $card(y, z)$ expresses that y and z have the same cardinality.

(4) Let $\psi = \forall y \exists z(\neg card(y, z) \wedge \exists x(inj(x, y, z))$. Is it true that $ZF \vdash \psi$?

Problem D For each of the following statements decide if it is true or false. Give a short explanation to support your answer.

(1) Let Γ be a set of sentences in some language L. Γ is maximally formally consistent if, and only if, for any two L-sentences φ, ψ holds that if $\Gamma \vdash \varphi \vee \psi$ then $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.

(2) Assume M is a model of ZF (Zermello-Fraenkel set theory). There is no proof tree with conclusion the sentence $\exists x \forall y(y \in x)$.

(3) Let L be a language and T an L-theory. If T admits quantifier elimination then T is complete.

(4) Let L be a language and $T \subseteq T'$ be L-theories. If T is complete then T' is complete.

(5) Consider $(\mathbb{R}, \leq)$, the real numbers with the usual order, and $X \subseteq \mathbb{R}$ a countable subset. Assume that with the induced order $(X, \leq)$ is a dense linear order. If $(X, \leq)$ has no smallest and no greatest element then $(X, \leq)$ is order isomorphich to $(\mathbb{Q} \cap (-\pi, \pi), \leq)$, the set of rational numbers between $-\pi$ and π with the induced order.

Grondslagen van de wiskunde Deeltentamen B (21-01-2010)
For the English text of this exam see the back of this page.

- Gebruik van dictaat of boeken is niet toegestaan. Gebruik van 1 A4 blad is wel toegestaan.

Opdracht A (Toepassing van de stelling van Loewenheim-Skolem)

(1) Zij L een taal met eindig veel symbolen. Gegeven zijn twee modellen M_1, M_2 van een L-theorie T en een L-zin φ zo dat $M_1 \models \varphi$ and $M_2 \models \neg\varphi$ geldt. Bewijs dat als M_1 en M_2 oneindig zijn dan bestaan er twee niet isomoorf T-modellen N_1 en N_2 zo dat $|N_1| = |N_2|$ geldt.

(2) Bewijs dat zonder de voorwaarde dat M_1 en M_2 oneindig zijn hoeft de conclusie niet waar te zijn.

Opdracht B Om te bewijzen dat $\vdash \varphi \leftrightarrow \neg\neg\varphi$ en $\vdash (\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$ moet men vier bewijsbomen geven. Uit deze vier kies er drie van en construeer ze.

Opdracht C Zij ZF de L-theorie van Zermelo-Fraenkel verzamelingenleer (dus $L = \{\in\}$). In onderdelen 1-3 wordt van je gevraagd een L-formule te geven die een bepaalde eigenschap van zijn variabelen beschrijft. Je hoeft niet te bewijzen dat de formule klopt. Let op dat je formules inderdaad L-formules zijn. In het geven van een formule mag je gebruik maken van formules uit eerdere onderdelen. Voor deze opdracht mag je gebruik maken van bekende feiten over verzamelingen uit het eerste deel van de cursus. Je hoeft niet een axiomatisch bewijs te geven.

(1) $func(x, y, z)$ zegt dat x een functie van y naar z is.

(2) $inj(x, y, z)$ zegt dat x is een injectieve functie van y naar z.

(3) $card(y, z)$ zegt dat y en z dezelfde kardinaliteit hebben.

(4) Zij $\psi = \forall y \exists z(\neg card(y, z) \wedge \exists x(inj(x, y, z))$. Klopt het dat $ZF \vdash \psi$?

Opdracht D Zijn de volgende uitspraken waar of onwaar? Geef voor elk van je antwoorden een kort argument.

(1) Zij Γ een verzameling van zinnen in een taal L. Γ is maximaal formeel consistent dan, en slechts dan als, voor elke twee L-zinnen φ, ψ geldt dat als $\Gamma \vdash \varphi \vee \psi$ dan moet $\Gamma \vdash \varphi$ of $\Gamma \vdash \psi$ waar zijn.

(2) Zij M een model van ZF (Zermello-Fraenkel set theory). Er bestaat geen bewijsboom met conclusie de zin $\exists x \forall y(y \in x)$.

(3) Zij L een taal en T een L-theorie. Als T kwantoreliminatie toelaat dan is T is volleedig.

(4) Zij L een taal en $T \subseteq T'$ twee L-theorieen. Als T volleedig is dan is T' volleedig.

(5) Beschouw $(\mathbb{R}, \leq)$, de reele getallen met de gewone orde structuur, en zij $X \subseteq \mathbb{R}$ een aftelbare deelverzameling. Stel voor dat $(X, \leq)$ met de geinduceerde orde een dichte lineaire orde is. Als $(X, \leq)$ noch kleinste noch grootste element heeft dan is $(X, \leq)$ orde isomoorf met $(\mathbb{Q} \cap (-\pi, \pi), \leq)$, de verzameling van rationale getallen tussen $-\pi$ and π met de geinduceerde orde.

Solutions

Problem A:

(1) Since L has finitely many symbols it follows easily (this is also an exercise in the Lecture Notes) that $||L|| = \omega$. Since it is given that both M_1 and M_2 are infinite, that is $|M_i| \geq \omega$, the Loewenheim-Skolem Theorem applies. That is given any cardinality $\kappa \geq \max(|M_1|, |M_2|)$ there exist T-model N_1 and N_2 of cardinality κ and moreover M_i and N_i satisfy the same L-sentences. Specifically then $N_1 \models \varphi$ and $N_2 \models \neg\varphi$. Since isomorphic models satisfy the same sentences (another exercise from the Lecture Notes) it follows that if $N_1 \cong N_2$ then $N_1 \models \varphi$ and $N_1 \models \neg\varphi$ but in a given model any sentence is either true or false and never both. Thus N_1 and N_2 are not isomorphic.

(2) Consider the empty language L and the theory T whose models are those sets having either one or two elements (such a theory can easily be given explicitly). Now let M_1 be a set with one element and M_2 one with two elements. Let φ a sentence that says that a model has exactly one element. Then $M_1 \models \varphi$ and $M_2 \models \neg\varphi$. However, there are no two non-isomorphic models with the same cardinality since if N_1 and N_2 are non-isomorphic models then neccesarily (without loss of generality) N_1 has one element and N_2 has two elements. But then they do not have the same cardinality.

Problem B:

All four proof trees needed appear in the sheet prepared by Jeroen Goudsmit.

Problem C:

(1) A function is a relation with a special property (as explained in the Lecture Notes). Thus

$$func(x, y, z) = rel(x, y, z) \wedge (\forall u \forall v \forall t((u, v) \in x \wedge (u, t) \in x) \rightarrow (v = t)))$$

where $rel(x, y, z)$ expresses the fact that x is a subset of $y \times z$, thus

$$rel(x, y, z) = \forall u(u \in x) \rightarrow u \in y \times z$$

and $y \times z$ is as defined in the Lecture Notes.

(2) $inj(x, y, z) = func(x, y, z) \wedge (\forall u \forall v \forall t((u, t) \in x \wedge (v, t) \in x) \rightarrow (v = u)))$.

(3) $card(y, z) = \exists x(inj(x, y, z) \wedge \exists x'(inj(x', z, y)$. $Card(y, z)$ holds precisely when there is an injection from y to z and an injection from z to y. From the Cantor-Berenstein Theorem it follows that then $|y| = |z|$.

(4) It is true that $ZF \vdash \psi$. To prove that it is enough, by the Completeness Theorem to prove that $ZF \models \psi$. Indeed ψ asserts that for every set y there exists a set z with a bigger cardinality then that of y. Cantor's Lemma establishes that this indeed holds.

Problem D:

(1) The statement is false. Let T be an inconsistent model in some language. Then $T \vdash \psi$ holds for all sentences ψ but T is not formally consistent and certainly not maximal formally consistent. (The other implication is true and is an exercise in the Lecture Notes. Mentioning this does earn some points).

(2) The statement is true. Let ψ be the sentence in question. It is not true that $ZF \vdash \psi$ since ψ asserts the existence of the set of all sets. For such a set holds $x \in x$ which we have seen is impossible in a model. If $ZF \vdash \psi$ then, by the Soundness Theorem it follows that $ZF \models \psi$, which is not the case.

(3) The statement is not true. A counter example is the theory T_{acf} of algebraicaly closed fields. We have seen that it admits quantifier elimination but is not complete.

(4) The statement is true. To prove that T' is complete consider an arbitrary L-sentence φ. If T is inconsistent then certainly $T \models \varphi$ (an inconsistent theory makes any sentece true). Otherwise let M be any model of T'. Now since T is complete we have either $T \models \varphi$ of $T \models \neg\varphi$. Assume $T \models \varphi$. Then since M is also a model of T we have $M \models \varphi$. M was arbitrary thus $T' \models \varphi$. A similar argument holds if $T \models \neg\varphi$.

(5) The statement is correct. We use the theorem that the theory of linear dense orders with no end-points is ω-categorical. The assumptions on $(X, \leq)$ say exactly that it is a countable dense linear order with no end-points. Thus it is order isomorphic to any other countable dense linear order with no end-points. The described set is exactly such an order (it is clearly countable since it's a subset of $\mathbb{Q}$ and it has no end-points since $\pi, -\pi$ are irrational).