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In 2008-2009, the course WISB324 was given by Dr. J. Stienstra.

Voorstellingen van eindige groepen (WISB324)
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During this exam you may consult the book “Representations and characters of groups” by James and Liebeck.
Do not only give answers to the exam problems, but also show clearly how you arrive at these answers.
In case you cannot answer some part of a problem, you may continue using the results.

Question 1

Let G be a finite group. Let V be a $\mathbb{C}G$-module and let χ be its character. Let U be an *irreducible* $\mathbb{C}G$-module and let ψ be its character. Let z denote the following element of the group algebra $\mathbb{C}G$:

$$z = \sum_{g \in G} \chi(g) g$$

a) Show that for every $h \in G$ we have: $\qquad hzh^{-1} = z$.

b) Define the map $\zeta : U \to U$ by $\zeta(u) = zu$ for every $u \in U$.
Show that ζ is a $\mathbb{C}G$-homomorphism.

c) Show that there is a number $\lambda \in \mathbb{C}$ such that $\zeta(u) = \lambda u$ for every $u \in U$.

d) Compute the number $\frac{1}{\lambda}\langle \overline{\chi}, \psi \rangle$.

Note: $\langle \overline{\chi}, \psi \rangle$ is the inner product of the characters $\overline{\chi}$ and ψ.
Hint: compute the trace of the linear map ζ in two ways.

P.T.O. / Z.O.Z.

Question 2

In this problem the group G is given by generators a, b, c and defining relations $a^3 = 1$, $b^3 = 1$, $c^2 = 1$, $ab = ba$, $ca = a^2c$, $cb = b^2c$; here 1 denotes the identity element of G.
It can be shown (but you do not have to do that here) that all elements of G can be written uniquely in the form $a^ib^jc^k$ with $i, j \in \{0, 1, 2\}$, $k \in \{0, 1\}$ and that the order of G is 18.

a) Show that the group G has 6 conjugacy classes $C_1, \ldots, C_6$ and give for each C_j all elements in that conjugacy class. *Remark:* you should find $1 \in C_1, a \in C_2, b \in C_3, ab \in C_4, a^2b \in C_5, c \in C_6$.

b) Show that there is a 1-dimensional representation of G with character χ satisfying $\chi(C_j) = 1$ for $j = 1, 2, 3, 4, 5$ and $\chi(C_6) = -1$.

c) Show that G has precisely four irreducible characters of degree 2.

d) Consider the elements $\alpha(123)$, $\beta = (456)$, $\gamma = (12)(45)$ in the permutation group S_6.
Show that there is a homomorphism of groups $\phi : G \to S_6$ such that $\phi(a) = \alpha, \quad \phi(b) = \beta, \quad \phi(c) = \gamma$.

e) Let V be a 6-dimensional $\mathbb{C}$-vectorspace with basis $e_1, e_2, e_3, e_4, e_5, e_6$.
Show that there is a representation ρ of G such that for every $g \in G$

$$\rho(g)e_j = e_{\phi(g)(j)} \quad \text{for } j = 1, \ldots 6$$

Note: $\phi(g)(j)$ is the image of j under the permutation $\phi(g)$ of $1, \ldots, 6$. Thus ρ is the restriction to G of the standard permutation representation of S_6.

f) Compute the character χ_ρ of the represeantation ρ.

g) Show that the 1-dimensional spaces $\mathbb{C}(e_1 + e_2 + e_3)$ and $\mathbb{C}(e_4 + e_5 + e_6)$ are $\mathbb{C}G$-submodules of V and compute their characters.

h) Let $W \subset V$ be the linear subspace spanned by the vectors $v_1 = e_1 + \omega e_2 + \omega^2 e_3$ and $v_2 = e_1 + \omega^2 e_2 + \omega e_3$ where $\omega = e^{2\pi i/3} \in \mathbb{C}$
Show that W is a $\mathbb{C}G$-submodule of V.

i) Show that the $\mathbb{C}G$-module W is irreducible.

j) Let χ_3 denote the character of the $\mathbb{C}G$-module W.
Compute $\chi_3(C_j)$ for $j = 1, \ldots, 6$.

k) Show that $\chi_\rho = 2\chi_1 + \chi_3 + \chi_4$ where χ_1 is the trivial character, χ_3 is the character of the $\mathbb{C}G$-module W and χ_4 is another irreducible character.

l) Give the character table of the group G.
Hint: From the above you already know four rows of the character table. Use the orthogonality relations to find the remaining irreducible characters.