Exam: Representations of finite groups (WISB324)

Wednesday June 29, 9.00-12.00 h.

- You are allowed to bring one piece of $A4$-paper, wich may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number $(\cdot)$ are worth 1 point, except for 1(f), 1(h), 2(e), 3(b) and 3(f) which are worth 2 points. Exercise 1(i) is a bonus exercise, which is worth 2 points.
- Do not only give answers, but also prove statements, for instance by refering to a theorem in the book.

Good luck.

1. Let G be a non-commutative group of order 8.
   (a) Show that there is no element of order 8.
   (b) Show that there are elements of G that have order 4.
   (c) Show that G has exactly 5 conjugacy classes and determine the degrees of the irreducible representations of G.

   Now let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group, satisfying the relations

   $$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j\,.$$

   (d) Determine all conjugacy classes of Q.
   (e) Show that $\langle i \rangle$ (the group generated by i) is a normal subgroup of Q.
   (f) Calculate the character table of Q.
   (g) Determine the character of the regular representation of Q.
   (h) Determine all normal subgroups of Q.
   (i) (Bonus exercise) Find explicitly the matrices in $GL(n, \mathbb{C})$ for all elements of the irreducible representation of Q for which n is maximal.

   Answers:
   (a) If G has an element of order 8, then $G = C_8$, the cyclic group of order 8, wich is abelian. Contradiction.
   (b) If G also has no element of order 4, then G has elements of order 1, the unit 1, and all other elements have order 2. Now let x and y be elements of order 2, then $x^{-1} = x$ and $y^{-1} = y$, thus $1 = (xy)^2 = xyxy$ and $yx = yxxyxy = yyxy = xy$. But in that case all elements commute and G is abelian. Contradiction.
   (c) Use the fact that the number of conjagucy classes is equal to the number of irreducible characters. Then using the following formula for the degrees of he characters:

   $$\sum_{i=1}^{n} d_j^2 = |G|,$$

and the fact that one of the modules is the trivial module of degree $d = 1$. Now not all degrees can be 1, since then the group would be abelian. Thus, the only possibilities for the degrees is $1, 1, 1, 1$ and 2, hence $n = 5$ and there are 5 conjugacy classes.
(d) $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$ and $\{\pm k\}$.
(e) $\langle i \rangle = \{i, -1, -i, 1\}$ is isomorphic to C_4. Clearly 1 and -1 commute with all elements. We only have to conjugate $\pm i$ with j and k.

$$ji(-j) = (-k)(-j) = -i, \quad ki(-k) = j(-k) = -i,$$

indeed $\langle i \rangle$ is normal.
N.B. $\langle j \rangle$ and $\langle k \rangle$ are also normal subgroups.
(f) If we take G/H for $H = \langle i \rangle$, $\langle j \rangle$ and $\langle k \rangle$ we obtain C_2, which is abelian and which has 2 irreducible characters the trivial one and $\chi(1) = 1$, $\chi(a) = -1$ here is a the generator of C_2. We can lift these characters to the group and thus get 4 of the 5 characters of G. The last one χ_5 we can then calculate using the orthogonality relations of the columns of the character table. We thus get:

	1	-1	i	j	k	
χ_1	1	1	1	1	1	
χ_2	1	1	1	-1	-1	lift of $\langle i \rangle$
χ_3	1	1	-1	1	-1	lift of $\langle j \rangle$
χ_4	1	1	-1	-1	1	lift of $\langle k \rangle$
χ_5	2	-2	0	0	0	

(g) $\chi_{regular} = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5$ and $\chi_{regular}(1) = 8$ and $\chi_{regular}(g) = 0$ for $g \neq 1$.
(h) All normal subgroups, except $\{1\}$ can be found as intersections of kernels of linear characters. All irreducible characters are linear, except χ_5. Thus we obtain G (kernel of χ_1), $\langle i \rangle$ (kernel of χ_2), $\langle j \rangle = \{\pm 1, \pm j\}$ (kernel of χ_3), $\langle k \rangle = \{\pm 1, \pm k\}$ (kernel of χ_4). Now taking intersections we only obtain $\{\pm 1\}$.
(i) Note that we only have to define $\pm i$ and $\pm j$ because they generate the whole group. The standard one for $a + bi + cj + dk$ for $a, b, c, d = 0, \pm 1$ is

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

2. Let $\mathbb{F} = \mathbb{C}$ and let G be a group.
(a) Let $x \in G$, show that $C_x = \sum_{g \in x^G} g$ is in the center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$.
(b) Show that $C_x = C_y$ if and only if $y \in x^G$.
(c) Let G have k conjugacy classes and let $x_1, x_2, \ldots, x_k$ be representatives of these different conjugacy classes. Show that $C_{x_1}, C_{x_2}, \cdots, C_{x_k}$ are linearly independent.
(d) Let $\chi_1, \chi_2, \ldots \chi_\ell$ be the collection of all irreducible characters of G, prove that $D_i = \sum_{g \in G} \chi_i(g^{-1}) g$ is in $Z(\mathbb{C}G)$.
(e) Prove that

$$\mathrm{span}(C_{x_1}, C_{x_2}, \ldots, C_{x_k}) = \mathrm{span}(D_1, D_2, \ldots, D_\ell).$$

(f) Prove that the elements D_i are also linearly independent.

Answers:
(a) Since we sum over a conjugacy class and $hx^gh^{-1} = x^G$, we have

$$hC_xh^{-1} = \sum_{g\in x^G} hgh^{-1} = \sum_{h^{-1}gh\in x^G} g = \sum_{g\in x^G} g,$$

thus $hC_x = C_xh$ for $h \in G$. This is not enough we have to prove that C_x is in the center of the group algebra. So let $r = \sum_{h\in G} \lambda_h h$, then

$$rC_x = \sum_{h\in G} \lambda_h hC_x = \sum_{h\in G} \lambda_h C_x h = C_x \sum_{h\in G} \lambda_h h = C_x r$$

and $C_x \in Z(\mathbb{C}G)$.
(b) Note that the conjugacy classes form a partition of G. Now, if $y \in x^G$, then $x^G = y^G$ and $C_x = C_y$. If, however, $y \notin x^G$, then $x^G \cap y^G = \emptyset$, hence $C_x \neq \mathbb{C}_y$.
(c) Since the conjugacy classes form a partition of G, we have

$$0 = \sum_{i=1}^{k} \lambda_k C_{x_k} = \sum_{i=1}^{k} \lambda_k \sum_{g\in_k^G} g = \sum_{g\in G} \lambda_g g,$$

where $\lambda_g = \lambda_i$ if $g \in x_i^G$. Since the elements g form a basis of $\mathbb{C}G$, we find that all $\lambda_g = 0$ and hence all $\lambda_i = 0$, which gives that the C_{x_i} are linearly independent.
(d) Note that $k = \ell$ since the number of irreducible characters is equal to the number of conjugacy classes of G and that characters are constant on conjugacy classes, hence $D_i = \sum_{j=1}^{k} \chi(x_j^{-1})C_{x_j}$ is a linear ombination of the elements $C_{x_j} \in Z(\mathbb{C}G)$. Thus $D_i \in Z(\mathbb{C}G)$.
(e) Note that $D_i = \sum_{g\in G} \overline{\chi_i(g)} g$ and that

$$(D_1, \dots D_k)^T = \overline{\chi}(C_{x_1}, \dots C_{x_k})^T,$$

where χ is the matrix of the character table. Since χ is invertible, so is $\overline{\chi}$. Thus

$$(C_{x_1}, \dots C_{x_k})^T = \overline{\chi}^{-1}(D_1, \dots D_k)^T$$

Which proves (e) but also (f).
(f) See (e).

3. Let $H \leq G$ and let χ be a character of H.
(a) Prove that $\chi \uparrow G(1) = [G : H]\chi(1)$.
(b) Which irreducible character of the Quaternion group Q of exercise 1 is induced by a character of one of its subgroups?
(c) Let H be in the center $Z(G)$ of G, prove that

$$\chi \uparrow G(g) = \begin{cases} [G : H]\chi(g) & if\ g \in H, \\ 0 & if\ g \notin H. \end{cases}$$

From now on let $G = D_{4n} = \langle a, b | a^{2n} = b^2 = 1,\ ab = ba^{-1} \rangle$.
(d) Determine the center $Z(D_{4n})$ of D_{4n}.
(e) Let $n \geq 2$, $H = Z(D_{4n})$ and χ be the non-trivial irreducible character of H, determine the values of $\chi \uparrow G(g)$ for $g \in D_{4n}$.
(f) The irreducible characters of $D_{4n}(\ n \geq 2)$ have the following values on 1 and a^n:

- $(\psi(1), \psi(a^n)) = (1, 1)$,
- $(\psi(1), \psi(a^n)) = (1, -1)$,
- $(\psi(1), \psi(a^n)) = (2, 2)$,
- $(\psi(1), \psi(a^n)) = (2, -2)$.

Determine in all 4 cases the multiplicity of ψ in $\chi \uparrow G$.

Answers:
(a) Use the answer of (c) or write $G = \cup_{1 \leq i \leq s} g_i H$ where this is a disjoint uniion, then $s = [G : H]$. Let V be the $\mathbb{C}H$-module that corresponds to χ, then the induced $\mathbb{C}G$-module is $\bigoplus_{i=1}^{s} g_i V$ hence its dimension is $s \dim(V) = [G : H] \dim(V) = \chi \uparrow G(1)$.
(b) We do not consider the case $G = H$ because that would not be induced from a proper subgroup. Using (a) the only possibility is the character that is not linear. Since $\chi_5(1) = 2$, H must be a subgroup of order 4, since $[G : H]$ must be 2. Take $H = \langle i \rangle$, and the character $\psi(i^k) = i^k$ The only conjugacy classes of G that have non-empty intersection with H are $\{1\}, \{-1\}$ and $\{\pm i\}$ Thus $\psi \uparrow G(\pm k) = \psi \uparrow G(\pm j) = 0$ and

$$\psi \uparrow G(\pm i) = 4 \left(\frac{\psi(i)}{4} + \frac{\psi(-i)}{4} \right) = 0, \qquad \psi \uparrow G(\pm 1) = 8 \frac{\psi(\pm 1)}{4} = \pm 2.$$

Hence $\psi \uparrow G = \chi_5$.
(c) We use the same formula as we have used in (b). Since H is the center of G every conjugacy class of an element $h \in H$ consists of only the element h (this both in G and in H). Thus

$$\chi \uparrow G(h) = |C_G(h)| \frac{\chi(h)}{|C_H(h)|} = [G : H] \chi(h) \quad \text{if } h \in H \quad \text{and } = 0 \quad \text{otherwise.}$$

(d) This is $\{1, a^n\}$.
(e) Apply the formula of (c) this gives the desired result:

$$\chi \uparrow G(g) = 0 \quad \text{for } g \neq 1, a, \qquad \chi \uparrow G(1) = 2n, \quad \chi \uparrow G(a^n) = -2n.$$

(f) The multiplicity of ψ in $\chi \uparrow G$ is equal to

$$\langle \chi \uparrow G, \psi \rangle_G = \frac{1}{|G|} \left(2n\psi(1) - 2n\psi(a^n) \right) = \frac{1}{2} (\psi(1) - \psi(a^n)),$$

which is 0,1,0,2, respectively.