

Exam: Representations of finite groups (WISB324)

Wednesday June 6, 9.00-12.00 h.

- You are allowed to bring one piece of A4-paper, which may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number (\cdot) are worth 1 point, except for 1(f), 1(h), 2(e), 3(b) and 3(f) which are worth 2 points. Exercise 1(i) is a bonus exercise, which is worth 2 points.
- Do not only give answers, but also prove statements, for instance by referring to a theorem in the book.

Good luck.

1. Let G be a non-commutative group of order 8.
 - (a) Show that there is no element of order 8.
 - (b) Show that there are elements of G that have order 4.
 - (c) Show that G has exactly 5 conjugacy classes and determine the degrees of the irreducible representations of G .

Now let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group, satisfying the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

- (d) Determine all conjugacy classes of Q .
 - (e) Show that $\langle i \rangle$ (the group generated by i) is a normal subgroup of Q .
 - (f) Calculate the character table of Q .
 - (g) Determine the character of the regular representation of Q .
 - (h) Determine all normal subgroups of Q .
 - (i) (Bonus exercise) Find explicitly the matrices in $GL(n, \mathbb{C})$ for all elements of the irreducible representation of Q for which n is maximal.
2. Let $\mathbb{F} = \mathbb{C}$ and let G be a group.
 - (a) Let $x \in G$, show that $C_x = \sum_{g \in xG} g$ is in the center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$.
 - (b) Show that $C_x = C_y$ if and only if $y \in xG$.
 - (c) Let G have k conjugacy classes and let x_1, x_2, \dots, x_k be representatives of these different conjugacy classes. Show that $C_{x_1}, C_{x_2}, \dots, C_{x_k}$ are linearly independent.
 - (d) Let $\chi_1, \chi_2, \dots, \chi_\ell$ be the collection of all irreducible characters of G , prove that $D_i = \sum_{g \in G} \chi_i(g^{-1})g$ is in $Z(\mathbb{C}G)$.
 - (e) Prove that

$$\text{span}(C_{x_1}, C_{x_2}, \dots, C_{x_k}) = \text{span}(D_1, D_2, \dots, D_\ell).$$

- (f) Prove that the elements D_i are also linearly independent.

3. Let $H \leq G$ and let χ be a character of H .
- (a) Prove that $\chi \uparrow G(1) = [G : H]\chi(1)$.
 - (b) Which irreducible character of the Quaternion group Q of exercise 1 is induced by a character of one of its subgroups?
 - (c) Let H be in the center $Z(G)$ of G , prove that

$$\chi \uparrow G(g) = \begin{cases} [G : H]\chi(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

From now on let $G = D_{4n} = \langle a, b \mid a^{2n} = b^2 = 1, ab = ba^{-1} \rangle$.

- (d) Determine the center $Z(D_{4n})$ of D_{4n} .
- (e) Let $H = Z(D_{4n})$ and let χ be the non-trivial irreducible character of H , determine the values of $\chi \uparrow G(g)$ for $g \in D_{4n}$.
- (f) The irreducible characters of D_{4n} have the following values on 1 and a^n :

- $(\psi(1), \psi(a^n)) = (1, 1)$,
- $(\psi(1), \psi(a^n)) = (1, -1)$,
- $(\psi(1), \psi(a^n)) = (2, 2)$,
- $(\psi(1), \psi(a^n)) = (2, -2)$.

Determine in all 4 cases the multiplicity of ψ in $\chi \uparrow G$.