Final Exam
Representation of Finite Groups
24.06.2020, 9:00-12:00

Important

You have to upload your solution until 12:30.
Students with extra time have 30 minutes more. If that is true for you, you have to upload until 13:00.

You have to add the following declaration to your solution:

Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van het internet.

There are 43 points to earn in the exam. 3 of those are bonus points. That means you receive the maximal grade if you get 40 or more points. You can use the statement of previous parts of a question even if you were not able to prove them.

Question 1

Let $G = \langle a, b : a^5 = b^4 = 1 \ , \ b^{-1}ab = a^3 \rangle$. It can be shown that G has 20 elements of the form $a^k b^i$ with $k \in \{0, \dots, 4\}$ and $i \in \{0, \dots 3\}$. Let H be the subgroup of G generated by a.

a) Show that H is a normal subgroup and that G/H is abelian. (2 points)

b) Determine the five conjugacy classes of G. (3 points)

c) Find all linear characters of G. (2 points)

d) Find the complete character table of G. (2 points)

e) Find all normal subgroups of G. (2 point)

Question 2

Let G be a finite group. In this question we will proof the following result:

Theorem 1. *The sum of all elements in any fixed row of the character table of G is a non negative integer.*

For this, we consider the action of G on the group algebra by conjugation, i.e. for $g \in G$, $h \in \mathbb{C}[G]$ we define

$$g * h = ghg^{-1}.$$

a) Show that the above multiplication makes $\mathbb{C}[G]$ into a $\mathbb{C}G$ module. (2 points)

b) Let χ_C be the character of this module. Show that $\chi_C(g) = |C_G(g)|$. (= The number of elements of the centralizer of g.) (2 points)

c) Use b) to show that the sum of elements in a row of the character table associated with a given character χ can be expressed as $\langle \chi, \chi_C \rangle$. (2 points)

d) Using c), complete the proof of the theorem. (2 points)

Question 3

Let $D_8 = \langle a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ and consider

$$V = \{ \sum_{i,j=0}^{3} \lambda_{i,j} x_i y_j : \lambda_{i,j} \in \mathbb{C} \}.$$

This is a 16 dimensional subspace of the vector space of polynomials, with basis $\{x_i y_j : i, j \in \{0, 1, 2, 3\}\}$. We define a group action on V by describing the multiplication of generators of G with this basis. We set

$$a(x_i y_j) = x_{i+1(4)} y_{j-1(4)},$$

where we denote by (4) that the index is calculated modulo 4, so $3 + 1 = 0(4)$ and $0 - 1 = 3(4)$. Further we set

$$b(x_i y_j) = x_j y_i.$$

Recall that you can find the character table for D_8 at 16.3(3) in the book.

a) Show that the above multiplication makes V to a D_8 module. (2 points)

b) Calculate the character of V, say χ. (2 points)

c) Decompose χ into irreducible characters. (3 points)

d) Find a basis for each of the submodules of V that is isomorphic to the the trivial module. (Hint: Use 14.26) (3 points)

e) Find a basis for all irreducible two dimensional submodules of V. (Hint: Use again 14.26 and then group the found vectors suitably into pairs.) (3 points)

Question 4

Let G be a finite group and $n \in \mathbb{N}$. Define for $g \in G$ the function

$$r_n(g) = |\{h \in G : h^n = g\}|.$$

That is, $r_n(g)$ counts the number of ways g can be expressed as an n-th power.

a) Show that $r_n(g)$ is constant on conjugacy classes. (So it is a so-called class function.) (2 points)

b) Deduce from a) that there are $a_\chi \in \mathbb{C}$ such that $r_n(g) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi(g)$, where the sum runs over all irreducible characters of G. (1 point)

c) Given an irreducible character χ of G, find an expression for a_χ. (3 points)

d) Let G be abelian and χ_1 denote the trivial character of G. Show that there exists a character of G, say ψ, such that $a_\chi = \langle \psi, \chi_1 \rangle$. (3 bonus points)

e) Conclude that, if G is abelian, $r_n(g)$ is a character of G. (2 points)

Solution 1

We have

$$\begin{aligned}
b^{-1}ab &= a^3 \\
b^{-1}a^3b &= a^4 \\
b^{-1}a^2b &= a^2 \\
b^{-1}a^6b &= a
\end{aligned}$$

and so $a^l b^j a^k (a^l b^j)^{-1} \in H$.

There is $\{1\}$ and by above $H - \{1\}$. Further,

$$a^{-1}b^{-1}a = a^{-1}b^{-1}abb^{-1} = a^2b^{-1}$$

and so $a^{-l}b^{-1}a^l = a^{2l}b^{-1}$. Since 2 is coprime to 5, $2l$ runs over all residues $\mod (5)$ and consequently this gives the conjugacy class Hb^{-1}. For b^{-2} we get

$$\begin{aligned}
a^{-1}b^{-2}a &= a^{-1}b^{-1}b^{-1}abbb^{-2} = a^{-1}b^{-1}a^3bb^{-2} \\
=&a^{-1}b^{-1}abb^{-1}abb^{-1}abb^{-2} = a^8b^{-2}.
\end{aligned}$$

In general, the same argument yields

$$a^{-1}b^{-l}a = a^{3^l-1}b^{-l}.$$

For $l \in \{1,2,3\}$ we have that 5 does not divide $3^l - 1$ and so this generates conjugacy classes of the form Hb, Hb^2, Hb^3. Thus G has 5 conjugacy classes.

To find all linear characters, we have two approaches: Either start with the consideration that $\rho(a) = \alpha$ and $\rho(b) = \beta$. Then $\alpha^5 = \beta^4 = 1$ and $\alpha = \alpha^3$ so $\alpha^2 = 1$. The only possibility for this is $\alpha = 1$. This gives exactly four linear representations. Alternatively, use that G/H is isomorphic to the cyclic subgroup generated by b. That subgroup has 4 linear characters that lift to G. We know that G has precisely 5 irreducible characters, and since it has 20 elements, there can not be more than those 4 linear characters.

We know that there are 5 irreducible characters. Call their dimensions d_i. Then

$$\sum_{i=1}^{5} d_i^2 = 20$$

and $d_1, \dots d_4 = 1$ and so $d_5 = \sqrt{20-4} = 4$. Together with the linear characters, we get the character table

	$\{1\}$	$H - \{1\}$	Hb	Hb^2	Hb^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

Here we deduced the entries of the bottom row by the column orthogonality relations.

Normal subgroups are exactly intersections of kernels of the characters. We have

$$\begin{aligned}
\ker(\chi_1) &= G \\
\ker(\chi_2) &= \ker(\chi_4) = H \\
\ker(\chi_3) &= H \cup Hb^2 \\
\ker(\chi_5) &= \{1\}.
\end{aligned}$$

This gives us the normal subgroups $\{1\}, H, H \cup Hb^2, G$.

Solution 2

We have

$$(g_1g_2) * h = (g_1g_2)h(g_1g_2)^{-1} = g_1(g_2gg_2^{-1})g_1^{-1} = g_1 * (g_2 * h).$$

For any pair g, h the product $g*h$ is precisely one element of the group. Thus, to calculate the character $\chi_C(g)$, we need again count fixed points, i.e. count how often $g * h = h$. Now, $g * h = ghg^{-1} = h$ if and only if $gh = hg$ so if and only if $h \in C_G(g)$. It follows that the number of fixed points and so the value of the character is equal to the size of the centralizer.

Let ψ be the character whose row we are considering and let $g_1, \dots, g_k$ be a complete system of representatives of all conjugacy classes. We have

$$\begin{aligned}\langle \psi, \chi_C \rangle &= \sum_{i=1}^{k} \frac{\psi(g_i)\overline{\chi_C(g_i)}}{|C_G(g_i)|} \\ &= \frac{1}{|G|}\sum_{i=1}^{k} \psi(g_i).\end{aligned}$$

This is precisely the sum of all elements in the row of the character table, since every conjugacy class appears once as an argument.

For any two character χ_1, χ_2 associated to modules V_1 and V_2, we have

$$\langle \chi_1, \chi_2 \rangle = \dim(\mathrm{Hom}(V_1, V_2)),$$

which is a non negative integer. Thus the same holds for $\langle \psi, \chi_C \rangle$ and this completes the proof.

Solution 3

Let B be the basis of V given by elements of the form x_iy_j with $i, j \in \{0, 1, 2, 3\}$. Since we calculate the index $\mod 4$, it is clear that $a^4(x_iy_j) = x_iy_j$. Furthermore $b^2(x_iy_j) = b(x_jy_i) = x_iy_j$. Finally

$$bab^{-1}(x_iy_j) = ba(x_jy_i) = b(x_{j+1}y_{i-1}) = x_{i-1}y_{j+1} = a^{-1}(x_iy_j).$$

Since all entries are 1 or 0, the trace is equal to the number of 1 on the diagonal, that is the number of fixed points of the permutation. We can find the conjugacy classes in 16.3(3) and observe that a and a^2 have no fixed points, since $i \pm 1, 2 \neq i \mod 4$. b fixes x_iy_j if and only if $i = j$ and so has 4 fixed points. $ab(x_iy_j) = a(x_jy_i) = (x_{j+1}y_{i-1})$ and this is a fixed points, if and only if $i = j + 1$, so there are again 4 fixed points. This gives the character values

	$\{1\}$	a^2	a	b	ab
χ	16	0	0	4	4

With this we calculate

$$\begin{aligned}\langle \chi, \chi_1 \rangle &= \frac{1}{8}(16 + 0 + 0 + 2 * 4 + 2 * 4) = 4 \\ \langle \chi, \chi_2 \rangle &= \frac{1}{8}(16 + 0 + 0 - 2 * 4 - 2 * 4) = 0 \\ \langle \chi, \chi_3 \rangle &= \frac{1}{8}(16 + 0 + 0 + 2 * 4 - 2 * 4) = 2 \\ \langle \chi, \chi_4 \rangle &= \frac{1}{8}(16 + 0 + 0 - 2 * 4 + 2 * 4) = 2 \\ \langle \chi, \chi_5 \rangle &= \frac{1}{8}(32 + 0 + 0 + 0 + 0) = 4\end{aligned}$$

To find basis for the spaces, we first consider the trivial submodule with character χ_1. We have then

$$\sum_{g \in G} \chi(g^{-1})g = 1 + a + a^2 + a^3 + b + ba + ba^2 + ba^3.$$

Applying this on $x_i y_j$ we get

$$\sum_{l=0}^{4} x_{i+l} y_{j-l} + \sum_{m=1}^{4} x_{j-m} y_{i+m}$$

In both sums, we get exactly those $x_{i'} y_{j'}$ that fulfill $i' + j' \equiv i + j(4)$. Thus, we see that the 16 basis elements are grouped into four groups of four elements whose sum each forms a basis for a trivial submodule:

$$\begin{aligned}
b_1 &= x_0 y_0 + x_1 y_3 + x_2 y_2 + x_3 y_1 \\
b_2 &= x_0 y_1 + x_1 y_0 + x_2 y_3 + x_3 y_2 \\
b_3 &= x_0 y_2 + x_1 y_1 + x_2 y_0 + x_3 y_3 \\
b_4 &= x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0.
\end{aligned}$$

For the two dimensional module belonging to χ_5, we observe

$$\sum_g \chi_5(g^{-1}) g = 2 - 2a^2.$$

Applying this on $\frac{1}{2} x_i y_j$, we get

$$x_i y_j - x_{i+2} y_{j-2}.$$

This results in the 8 vectors

$$\begin{aligned}
&x_0 y_0 - x_2 y_2 && x_3 y_1 - x_1 y_3 \\
&x_1 y_0 - x_3 y_2 && x_0 y_1 - x_2 y_3 \\
&x_2 y_0 - x_0 y_2 && x_1 y_1 - x_3 y_3 \\
&x_3 y_0 - x_1 y_2 && x_2 y_1 - x_3 y_3
\end{aligned}$$

Further, multiplication of a or b on each of these vectors, either fixes it or send it to a scalar multiple of the other vector in the same row. Thus, each row forms a basis for the four copies of the irreducible two dimensional submodule of V.

Solution 4

We have $h^n = g$ if and only if $(k^{-1} h k)^n = k^{-1} h^n k = k^{-1} g k$ and so there is a one to one correspondence of representations of g as n-th powers and representations of its conjugates as n-th powers.

We know that the irreducible characters form a basis of the space of class functions, seen as a complex vector space. As such every class function is a linear combination of irreducible characters. (See corollary 15.4)

We observe that

$$r_n(g) = \sum_{h^n = g} 1.$$

Since the irreducible character form an ONB, we can find a_χ by the use of the inner product:

$$\begin{aligned}
a_\chi &= \langle r_n, \chi \rangle \\
&= \frac{1}{|G|} \sum_{g \in G} r_n(g) \overline{\chi}(g) \\
&= \sum_{h \in G} \overline{\chi}(h^n).
\end{aligned}$$

If G is abelian, then all irreducible character are linear. For linear characters we have $\overline{\chi}(h^n) = \overline{\chi}(h)^n$. We know from the book that powers of characters are characters and that complex conjugates of characters are characters, so $\overline{\chi}(h)^n$ is a character of G. This means that its inner product with the trivial character χ_1 is an integer (see above dimension argument for example) and so

$$a_\chi = \langle \overline{\chi}^n, \chi_1 \rangle$$

is a non negative integer. So all coefficients in a_χ are as such and thus $r_n(g)$ is the character to the module that consists of a direct sum over all irreducible χ and for each a_χ copies of the module associated to χ^n (see end of chapter 17 for the construction, since the characters are linear).