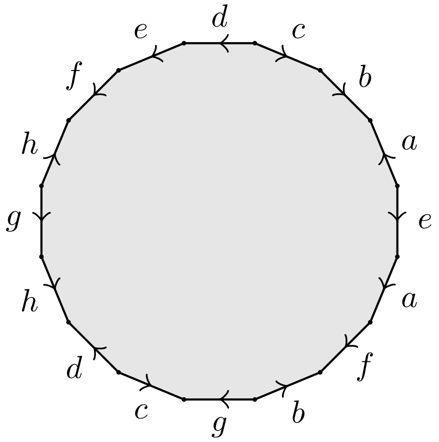
Midterm exam *Topologie en Meetkunde* (WISB341). A. Henriques, Mar 2012.
Do not simply provide answers: justify all your assertions.

Problem 1 State the definition of a manifold. **[2pt]** [1pt]
Prove that if M and N are manifolds, then their product $M \times N$ is also a manifold. [1pt]

Solution: A manifold a space X such that every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$. Given a point $(x, y) \in M \times N$, pick neighborhoods $U \subset M$ of x and $V \subset N$ of y such that $U \cong \mathbb{R}^m$ and $V \cong \mathbb{R}^n$. Then $U \times V \cong \mathbb{R}^{m+n}$ is a neighborhood of (x, y).

Problem 2 State the classification theorem for compact surfaces. **[3pt]** [1pt]
Let Σ be the surface obtained by glueing the sides of a regular 16-gon according to the following pattern:



To which surface in the classification is Σ homeomorphic? [2pt]

Solution: A compact surface is homeomorphic to either S^2, a connected sum of copies of T^2, or a connected sum of copies of P^2, and those are pairwise non-homeomorphic. The surface Σ is orientable and has Euler characteristic given by $3 - 8 + 1 = -4$. It is therefore homeomorphic to $T^2 \# T^2 \# T^2$.

Problem 3 Given two natural numbers $m < n$, the product $S^m \times S^n$ of the m-dimensional sphere with the n-dimensional sphere is a CW-complex with four cells. **[3pt]**
What are the dimensions of those cells? [1pt]
Describe the m-skeleton of that CW complex. [1pt]
Describe the n-skeleton of that CW complex. [1pt]

Solution: The cells have dimenions 0, m, n, and $m + n$. The m-skeleton is S^m, and the n-skeleton is $S^n \vee S^m$.

Problem 4 State the definition of homotopy equivalence. **[2pt]** [1pt]
Prove that if X and Y are two spaces that are homotopy equivalent, then the products $X \times S^1$ and $Y \times S^1$ are also homotopy equivalent. [1pt]

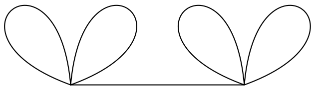
Solution: $X \approx Y$ if there are maps $f : X \to Y$, $g : Y \to X$, such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. The maps $f \times 1_{S^1} : X \times S^1 \to Y \times S^1$, and $g \times 1_{S^1} : Y \times S^1 \to X \times S^1$ satisfy $(f \times 1_{S^1}) \circ (g \times 1_{S^1}) \sim 1_{Y \times S^1}$ and $(g \times 1_{S^1}) \circ (f \times 1_{S^1}) \sim 1_{X \times S^1}$, and thus form a homotopy equivalence between $X \times S^1$ and $Y \times S^1$.

Problem 5 Consider a triangulation of $T^2 \# T^2$ such that at every vertex, exactly seven triangles meet. How many triangles are there in total in that triangulation? **[3pt]**

Solution: The Euler characteristic $F - E + V$ is equal to $\chi(T^2 \# T^2) = -2$. We have $E = \frac{3}{2}F$ and $V = \frac{3}{7}F$, therefore $\chi(T^2 \# T^2) = F(1 - \frac{3}{2} + \frac{3}{7}) \Rightarrow F = (-2)/(1 - \frac{3}{2} + \frac{3}{7}) = 28$.

Problem 6 The surface $T^2 \# T^2$ admits a CW complex structure whose 1-skeleton is the following graph:

Γ :



Describe an attaching map $f : S^1 \to \Gamma$ such that $\Gamma \cup_f e_2 = T^2 \# T^2$. **[2pt]**

Solution: If the 1-skeleton was $S^1 \vee S^1 \vee S^1 \vee S^1$, then the attaching map would be given by the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Calling the extra horizontal edge in Γ by the letter e, the attaching map is then given by $aba^{-1}b^{-1}ecdc^{-1}d^{-1}e^{-1}$.

Questions for the oral exam of *Topologie en Meetkunde*. June 2012.

Each students picks three questions at random, out of which he/she selects two that are then presented at the board. The student has 30 minutes to prepare, and 30 minutes for the presentation.

- Define the notion of orientability. Illustrate with examples of one, two, and three dimensional manifolds.
- State the classification theorem for surfaces, and explain the main steps of its proof.
- Define the notion of CW-complex, with particular emphasis on the topology. Illustrate with one and two dimensional examples.
- Define the notion of homotopy equivalence and illustrate it with a couple of examples. Prove that it is an equivalence relation.
- Define the homotopy group $\pi_n(X)$ of a topological space X and prove that they are invariant under homotopy equivalences.
- Explain how to read off the fundamental group of a CW-complex, and sketch the proof. Illustrate with some examples.
- State the cellular aproximation theorem, and use it to prove that the inclusion of the $(n+1)$-skeleton of some space into the whole space induces an isomorphism of n-th homotopy groups.
- Define the notion of cover of a topological space, and prove that covers satisfy the path lifting property.
- Define universal covers and explain their relation to the fundamental group. Illustrate with examples.
- Define the homology groups of a topological space (the definition depends on the fact that $d \circ d = 0$; prove that fact), and illustrate with examples.