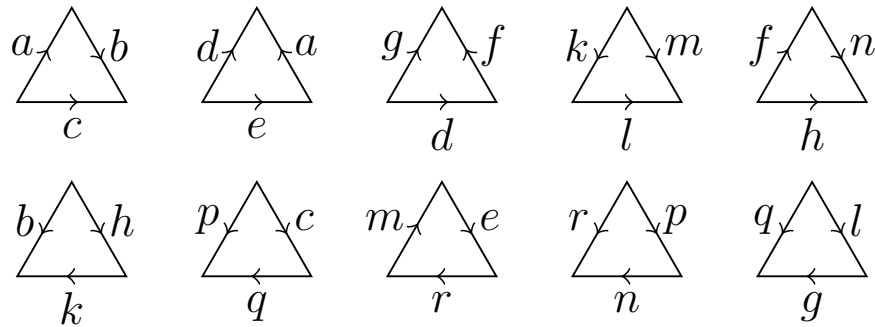Do not simply provide answers: justify all your assertions.

**Problem 1** State the classification theorem for compact 2-dimensional manifolds [4pt] [1pt] (no boundary).

Consider the 2-dimensional manifold built by glueing together the following ten triangles:



What is the Euler characteristic of that manifold?    [1pt]
Is that manifold orientable?    [1pt]
To which manifold in the classification is the above manifold homeomorphic?    [1pt]

**Solution:** Every compact connected surface is homeomorphic to $S^2$, a connected sum of copies of $P^2$, or a connected sum of copies of $T^2$. Moreover, the surfaces in the above list are pairwise non-homeomorphic. The manifold above has 10 triangles, 15 edges, and 6 vertices. Its Euler characteristc is therefore equal to $10 - 15 + 6 = 1$. The only manifold in the classification that has Euler characteristic 1 is $P^2$, which is not orientable. That manifold is therefore not orientable.

**Problem 2** Let $X$ be a topological space.    [4pt]
Define the group $\pi_2(X)$ (make sure to explain what the underlying set of $\pi_2(X)$ [3pt] is, what the product map $\pi_2(X) \times \pi_2(X) \to \pi_2(X)$ is, and what the inverse map $\pi_2(X) \to \pi_2(X)$ is.)
Prove that the group $\pi_2(X)$ is commutative.    [1pt]

**Solution:** $\pi_2(X)$ is the set of homotopy classes of maps $f : [0,1]^2 \to X$ that send $\partial[0,1]^2$ to the base point of $X$, where the homotopies are taken relatively to $\partial[0,1]^2$. The product of $[f]$ and $[g]$ is defined by cutting $[0,1]^2$ into two halves, sending the first half to $X$ by a suitably reparametrized version of $f$, and sending the second half to $X$ by a suitably reparametrized version of $g$. The inverse of $[f]$ is given by precomposing the map $f$ by the map $(x,y) \mapsto (-x,y)$. The product is commutative because one can continuously deform $fg$ into $gf$: the deformation involves first shrinking the two halves of $[0,1]^2$, then letting one go around the other one, and finally bringing them back to their original size. At any given stage through the deformation process, the two small rectangles are sent to $X$ via suitably reparametrized version of $f$ and $g$, while the complement goes to the base point.

**Problem 3** Define what it means for two spaces to be homotopy equivalent. [3pt] [1pt]
Show that the following two spaces are homotopy equivalent: [1pt]
  – The sphere $S^2$ minus $n$ points.
  – The wedge of $n-1$ copies of $S^1$.
Show that the following two spaces are not homotopy equivalent: [1pt]
  – The Möbius band.
  – The projective plane $P^2$.

**Solution:** Two spaces $X$ and $Y$ are homotopy equivalent is there exist maps $X \to Y$ and $Y \to X$ such that the composites $X \to Y \to X$ and $Y \to X \to Y$ are homotopic to $\mathrm{Id}_X$ and $\mathrm{Id}_Y$, respectively. $S^2$ minus $n$ points is homeomorphic to $\mathbb{R}^2$ minus $n-1$ points, which deformation retracts onto a wedge of $n-1$ copies of $S^1$. The fundamental group of the Möbius band is $\mathbb{Z}$, while that of the projective plane is $\mathbb{Z}/2$. The two spaces are therefore not homotopy equivalent.

**Problem 4** Pick a triangulation of the torus, describe the corresponding chain com- [3pt]
plex, and use it to compute the homology groups of $T^2$.

**Solution:** One can take a triangulation with two triangles, three edges and one vertex. The chain complex is $0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}^2 \leftarrow 0$. The first boundary map $\mathbb{Z} \leftarrow \mathbb{Z}^3$ is zero, from it follows that $H_0(T^2) = \mathbb{Z}$. The second boundary map $\mathbb{Z}^3 \leftarrow \mathbb{Z}^2$ is given by $(a,b) \mapsto (a-b, a-b, a-b)$. Its kernel is one dimensional, from which we conclude that $H_2(T^2) = \mathbb{Z}$. Its image is the set $\{(x,x,x)|x \in \mathbb{Z}\}$, and $H_1(T^2) = \mathbb{Z}^3/\{(x,x,x)\} \cong \mathbb{Z}^2$.

**Problem 5** Let $X := S^2 \vee T^2$. Write down $X$ as a $CW$-complex, describe the attach- [3pt]
ing maps, and use them to compute its fundamental group. Describe the universal cover of $X$.

**Solution:** $X$ has one 0-cell, two 1-cells (call them), and two 2-cells. The attaching map of the first 2-cell is given by $aba^{-1}b^{-1}$, while the attaching map of the second 2-cell is the constant map onto the base point. The fundamental group is given by $\langle a, b|aba^{-1}b^{-1}, e \rangle = \langle a, b|aba^{-1}b^{-1} \rangle = \mathbb{Z}^2$. The universal cover is $\mathbb{R}^2$ with a sphere attached to it at every point with integral coordinates.

**Problem 6** Construct a space whose first homotopy group is isomorphic to the group [3pt]
$(\mathbb{Q}, +)$ of rational numbers.

**Solution:** $\langle a, b, c, d, e, f, g, h.... |a = b^2, b = c^3, c = d^4, d = e^5, \ldots \rangle$ is a presentation of $(\mathbb{Q}, +)$. To turn that presentation into a space whose first homotopy group is $\mathbb{Q}$, take an infinite wedge of copies of $S^1$, and glue 2-cells via the attachings maps $ab^{-2}$, $bc^{-3}$, $cd^{-4}$, $de^{-5}, \ldots$