Geometry and Topology – Exam 2

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.0 pt)**.** Show that if a path connected and locally path connected space X has finite fundamental group then every map $X \longrightarrow S^1$ is null homotopic.

Exercise 2 (2.0 pt)**.** Let $\tilde{X}$ and $\tilde{Y}$ be path connected and simply connected covering spaces of the path connected and locally path connected spaces X and Y, respectively. Show that if X and Y are homotopy equivalent, then $\tilde{X}$ and $\tilde{Y}$ are also homotopy equivalent.

Exercise 3 (1.0 pt)**.** Given a map $f : S^{2n} \longrightarrow S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map from $\mathbb{R}P^{2n} \longrightarrow \mathbb{R}P^{2n}$ has a fixed point.

Exercise 4 (1.0 pt)**.** Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all degrees but their universal covering spaces do not.

Exercise 5 (1.0 pt)**.** Let X be the quotient space of S^2 under the identification $x \sim -x$ for x in the equator S^1. Compute all the homology groups of X.

Exercise 6 (2.0 pt)**.**

1. If X is a finite CW complex and $p : \tilde{X} \longrightarrow X$ is an n-sheeted covering map, then the Euler characteristic of X and $\tilde{X}$ are related by $\chi_{\tilde{X}} = n\chi_X$.
2. For every positive integer g, we let Σ_g be the closed oriented surface of genus g, i.e., $\Sigma_g = \#gT^2$. Show that if $p : \Sigma_g \longrightarrow \Sigma_h$, is a covering map, then $g = 1 \mod h - 1$.

Exercise 7 (2.0 pt)**.** Suppose that X is the union of open sets $A_1, \cdots, A_n$ such that for any subset $\{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$ the intersection

$$A_{i_1} \cap \cdots \cap A_{i_k}$$

is either empty or has trivial reduced homology groups. Show that $\tilde{H}_i(X) = \{0\}$ for $i \geq n - 1$.