Geometry and Topology – Exam 3

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.5 pt)**.** In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a)
$$\mathbb{K}\#T^2, \qquad \mathbb{R}P^2\#T^2\#S^2, \qquad \mathbb{K}\#\mathbb{R}P^2\#\mathbb{R}P^2, \qquad \mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2.$$

where $\mathbb{K}$ denotes the Klein bottle and T^2 is the 2-dimensional torus.

b)
$$S^{2n}, \qquad \mathbb{R}P^{2n}, \qquad \mathbb{C}P^n, \qquad \text{for } n > 1.$$

Exercise 2 (1.0 pt)**.** A deformation retraction in the weak sense of a space X to a subspace $A \subset X$ is a homotopy $f_t : X \longrightarrow X$ such that $f_0 = \mathrm{Id}$, $f_1(X) \subset A$ and $f_t(A) \subset A$ for all $t \in [0,1]$. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Exercise 3 (1.0 pt)**.** Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Exercise 4 (2.0 pt)**.** Let X and Y be path connected and locally path connected and let $\tilde{X}$ and $\tilde{Y}$ be their simply-connected covering spaces, respectively. Show that if X is homotopy equivalent to Y then $\tilde{X}$ is homotopy equivalent to $\tilde{Y}$.

Exercise 5 (1.0 pt)**.** Show that if $A \subset X$ is a retract of X then the map $H_n(A) \longrightarrow H_n(X)$ induced by the inclusion $A \hookrightarrow X$ is injective for all n.

Exercise 6 (1.5 pt)**.**

a) Given a map $f : S^{2n} \longrightarrow S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$.

b) Show that every map $f : \mathbb{R}P^{2n} \longrightarrow \mathbb{R}P^{2n}$ has a fixed point.

c) Construct a map $f : \mathbb{R}P^{2n-1} \longrightarrow \mathbb{R}P^{2n-1}$ without fixed points. (Hint: use linear a transformation $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ without eigenvectors.

Exercise 7 (2.0 pt)**.** Show that $H_i(X \times S^n) = H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ by definition. Hint: show $H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n; X \times \{x_0\})$ and $H_i(X \times S^n; X \times \{x_0\}) = H_{i-1}(X \times S^{n-1}; X \times \{x_0\})$.