Topology and Geometry B, Retake (August 27, 2010)

Note: Please motivate/prove each of your answers.

Exercise 1. Let T be the torus. Is it true that for any topological space X for which there exists a continuous bijection $f : X \longrightarrow T$, the fundamental group of X is isomorphic to $\mathbb{Z}^2$? (*1p*)

Exercise 2. Let A be a closed subset of a topological space X and let $r : X \longrightarrow A$ be a continuous map. Consider the statements:

(i) r is a retraction.

(ii) For all $a \in X$, $r_* : \pi(X, a) \longrightarrow \pi(A, a)$ is injective.

Which of the implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ holds true? (*2p*)

Exercise 3. Let $X = \mathbb{R}^2 - \{(0, 0)\}$, $x = (1, 0) \in X$ and consider

$$\gamma_1, \gamma_2 : [0, 1] \longrightarrow X$$

$$\gamma_1(t) = (cos(4\pi t), 2sin(4\pi t)),\ \ \gamma_2(t) = (cos(4\pi t), (2t-1)sin(4\pi t)).$$

Show that:

(i) γ_1 is homotopic to a constant map but γ_1 is not path-homotopic to the constant path. (*2p*)

(ii) γ_2 is path-homotopic to the constant path. (*1p*)

Exercise 4. Let A be the one-dimensional space from Figure 1. Consider also the space X which is the connected sum of a Moebius band and a torus (Figure 2).

(i) Compute the fundamental group of A and show the generators on the pictures. (*1p*)

(ii) Show how one can obtain X from a disk by gluing some of the points on the boundary of the disk. (*1p*)

(iii) Compute the Euler characteristic of X. (*1p*)

(iv) Compute the fundamental group of X. (*1p*)

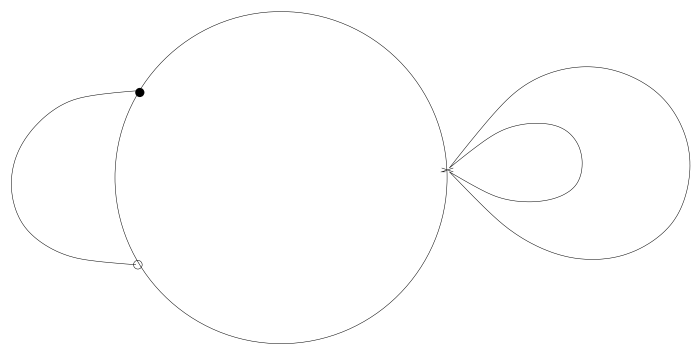

Figure 1:

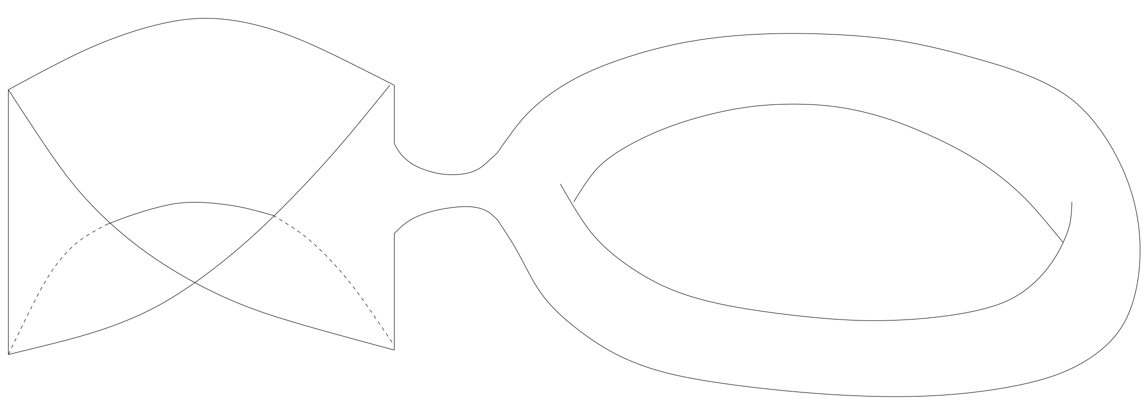

Figure 2: