Geometry and Topology – Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a)
$$T^2\#T^2, \qquad S^1\times S^1\times S^1\times S^1, \qquad T^2\#\mathbb{R}P^2\#\mathbb{R}P^2, \qquad ,\mathbb{K}\#\mathbb{K}.$$
where $T^2 = S^1\times S^1$ denotes the 2-torus and $\mathbb{K}$ denotes the Klein bottle.

b)
$$T^2\backslash\{p\}, \qquad S^1\vee S^1, \qquad \mathbb{K}\backslash\{p\}, \qquad S^2\backslash\{p_1,p_2,p_3\},$$

Exercise 2 (1.0 pt)**.** A *deformation retract in the weak sense* of a space X onto a subspace $A\subset X$ is a homotopy $F\colon I\times X\to X$ such that

- $F(0,x)=x,\ \forall x\in X,$
- $F(t,x)\in A,\ \forall x\in A,$
- $F(1,x)\in A,\ \forall x\in X.$

Show that if X deform retracts onto $A\subset X$ in the weak sense, the inclusion map $\iota\colon A\to X$ is a homotopy equivalence.

Exercise 3 (1.0 pt)**.** Show that a homotopy equivalence $f\colon X\to Y$ induces a bijection between the set of path components of X and the set of path components of Y and that f restricts to a homotopy equivalence from each path component of X to the corresponding path component of Y.

Exercise 4 (2.0 pt)**.** We can regard $\pi_1(X,x_0)$ as the set of base point preserving homotopy classes of maps $(S^1,s_0)\to(X,x_0)$. Let $[S^1,X]$ be the set of homotopy classes of maps $S^1\to X$ without conditions on basepoints. Thus there is a natural map $\Phi\colon\pi_1(X,x_0)\to[S^1,X]$ obtained by ignoring basepoints. Show that

a) If X is path connected, then Φ is onto.

b) If $f, g\colon (S^1, s_0) \to (X, x_0)$, then $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

c) Conclude that if X is path connected, $[S^1, X]$ is in bijection with the set of conjugacy classes in $\pi_1(X, x_0)$.

Exercise 5 (2.0 pt)**.** Compute the fundamental group of the following spaces.

a) The quotient space of S^2 obtained by identifying the north and south poles to a single point.

b) The quotient space of the disjoint union of two 2-tori obtained by identifying the circle $\{x_0\} \times S^1$ in one torus with the same circle in the other torus

Exercise 6 (2.0 pt)**.** Let $\tilde{X} \to X$ and $\tilde{Y} \to Y$ be simply connected covering spaces of X and Y. Show that if X and Y are path connected, locally path connected and $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$.