Geometry and Topology – Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** For each of the lists below, decide which spaces are homotopy equivalent to each other

- $\mathbb{R}^3\backslash\{0\}$, $\mathbb{R}^4\backslash\{0\}$ and $\mathbb{R}^5\backslash\{0\}$.
- $\mathbb{C}P^n$, S^{2n} and $\mathbb{R}P^{2n}$

Exercise 2 (2.0 pt)**.** Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Exercise 3 (2.0 pt)**.** Let $p : \tilde{X} \to X$ be a simply-connected covering space of X and let $A \subset X$ be a path-connected, locally path-connected subspace, with $\tilde{A} \subset \tilde{X}$ a path-component of $p^{-1}(A)$. Show that $p : \tilde{A} \to A$ is the covering space corresponding to the kernel of the map $\iota_* : \pi_1(A) \to \pi_1(X)$.

Exercise 4 (1.0 pt)**.** Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$, i.e., $F : D^n \to \mathbb{R}^n$, such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Exercise 5 (1.0 pt)**.** A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even* map. Show that an even map $f : S^n \to S^n$ must have zero degree if n is even. Hint: You can use without proof that

$$H_n(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } n \text{ odd} \\ \{0\} & \text{for } n \text{ even.} \end{cases}$$

Exercise 6 (2.0 pt)**.** Let $\mathcal{U} = \{U_1, \cdots U_k\}$ be an open cover of a space X with the following properties

- All the intersections of the form $U_{i_0} \cap \cdots \cap U_{i_l}$ are either contractible or empty (in particular, each U_i is contractible);
- There is an $n > 0$ for which $U_{i_0} \cap \cdots \cap U_{i_n} = \emptyset$ for all possible choices of distinct indices.

Show that $H_i(X) = \{0\}$ for all $i \geq n$.