Topologie en meetkunde – Final exam

- **Write your name and student number clearly on this exam.**
- You can give solutions in English or Dutch.
- You are expected to explain your answers.
- You are allowed to use results of the lectures, the exercises and homework.
- All maps in the statements of the problems are meant to be continuous.
- Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

The following is a reminder of definitions.

- A map $X \to Y$ is called *nullhomotopic* if it is homotopic to a constant map.
- A *surface* is a 2-dimensional compact connected manifold (without boundary).

Problem 1 (6 points)**.** *Show that a space X is homotopy equivalent to a point if and only if* $\mathrm{id}_X\colon X \to X$ *is nullhomotopic.*

Problem 2 (10 points)**.** *Show that there is no map $f\colon S^2 \to S^1$ whose restriction to the equator $S^1 \subset S^2$ is the identity.*

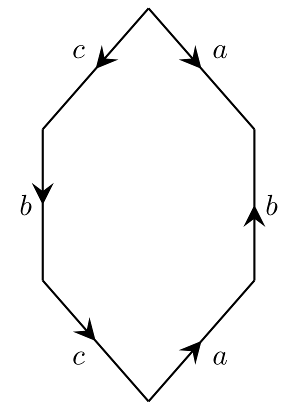
Problem 3 (20 points)**.** *Call a map* $f\colon S^1 \to S^1$ antipode-preserving *if it satisfies* $f(1) = 1$ *and* $f(-z) = -f(z)$*, where we view* S^1 *as a subset of* $\mathbb{C}$*.*

(a) Give for every odd number k *an example of an antipode-preserving map* f *such that* $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ *is multiplication by* k*.*

(b) Show that for every antipode-preserving map f*, the induced homomorphism* $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ *is multiplication by an odd number* $k \in \mathbb{Z}$*.*

Page for continuation of problem 3:

Problem 4 (12 points). *Define a surface S by glueing sides of a hexagon in the pattern depicted below. If $X_{m,n}$ is a surface obtained by attaching m cross-caps and n handles to a triangulated sphere, give all values of m and n such that S is homeomorphic to $X_{m,n}$.*



Problem 5 (12 points)**.** *Show that there is a map* $\gamma\colon S^1 \to \mathbb{C} \setminus \{1,2\}$*, which is not null-homotopic, but such it becomes nullhomotopic for every* $j \in \{1,2\}$ *if viewed as a map* $S^1 \to \mathbb{C} \setminus \{j\}$.

Problem 6 (20 points)**.** *Show that every map* $S^2 \to S^1$ *and every map* $\mathbb{RP}^2 \to S^1$ *are nullhomotopic. Give further an example (with proof) of a surface* S *with a map* $S \to S^1$ *that is not nullhomotopic.*

Page for continuation of problem 6:

Problem 7 (20 points)**.** *Let $X \to \mathbb{RP}^n \times \mathbb{RP}^n$ a covering map with X path-connected. For which $n \geq 1$ must X be necessarily compact? Give in each case a proof or a counterexample.*

Page for continuation of problems: