Final exam

Topologie en Meetkunde, Block 3, 2021

Instructions

General:

- The exam is 3 hours long (unless you are entitled to extra time).
- The exam is open book: you may use Hatcher or the lecture notes as a reference while working on it. However, you must sign and add to your exam the following declaration: "Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van andere hulpmiddelen dan het cursusboek/dictaat/eigen aantekeningen". This is particularly important if you are doing it from home.
- You have to justify all the claims you make. You may use results from the lectures or the book, but you should spell out what the hypothesis of the result are, and how they are satisfied.
- Try to be precise/clear: only one claim per sentence, only one key idea per paragraph.
- I recommend that you draw the spaces that appear in each of the exercises.

For those taking the exam online:

- You must keep Teams open and your camera on for the duration of the exam. Once the time is up, you will have 15 minutes (still with the camera on) in order to submit the exam through Blackboard.
- You must submit a PDF. You may produce this PDF either using Latex or by scanning a handwritten document. If you do the later, use a high quality scanner. There are many phone apps that do this.
- I will be available through Microsoft Teams during the exam. You may send me a private message if you have any questions.
- After the exam is complete, store your physical copy in case it needs to be given to the University for archiving.
- I may arrange a little chat after the exam with some of you to go over what you wrote.

Questions

Exercise 1 (1,5 points). Let $X \subset \mathbb{R}^n$ convex. Then:

- Compute $\pi_1(X, p, q)$, for every $p, q \in X$. You must do this explicitly from first principles (i.e. using only the definition of $\pi_1(X, p, q)$).
- Describe the fundamental groupoid $\Pi_1(X)$ (i.e. what is the composition law, what are the identities, what are the inverses, what is its topology). Check that all the maps are indeed continuous.

Exercise 2 (1 point). Construct a space with fundamental group

$$G := \langle g_1, \cdots, g_5 | g_1^7, g_2, g_3^5, g_4 g_5 g_4^{-1} g_5^{-1} \rangle.$$

Is there a compact surface without boundary whose fundamental group is G?

Exercise 3 (1,25 points). Let $p \in \mathbb{RP}^2$. Denote $A := \mathbb{RP}^2 \setminus \{p\}$ and $i : A \to \mathbb{RP}^2$ the inclusion.

- Let $x \in A$. Describe the fundamental groups of A and $\mathbb{RP}^2$ at x.
- Describe the pushforward $i_* : \pi_1(A, x) \to \pi_1(\mathbb{RP}^2, x)$.
- Is there a non nullhomotopic map $f : \mathbb{S}^1 \to A$ such that $i \circ f$ is nullhomotopic?

You can choose p and x to be placed in whatever manner is convenient for you.

Exercise 4 (1,25 points). Consider T^2 and let $C \subset T^2$ be a small disc. You may assume that T^2 is given by its standard planar representation as a square and C is a small standard disc in the interior of the square. Show that there is no retraction

$$r : (T^2 \setminus \mathring{C}) \to \partial C.$$

Exercise 5 (2 points). Let S be the 2-dimensional CW-complex with planar representation as in the Figure below.

- Is it a path-connected, compact surface without boundary?
- Is it orientable?
- Compute its Euler characteristic. Use this to determine all the g and g' such that S is homeomorphic to Σ_g or $N_{g'}$.
- Compute the fundamental group and its abelianisation. Use this to confirm your previous claim about S being homeomorphic to Σ_g or $N_{g'}$.
- Use a cut and paste argument to simplify the planar representation in order to obtain the claimed Σ_g or $N_{g'}$.
- Let $X_{m,n} := (\#_m \mathbb{RP}^2)\#(\#_n T^2)$. Determine those m, n such that S is homeomorphic to $X_{m,n}$.

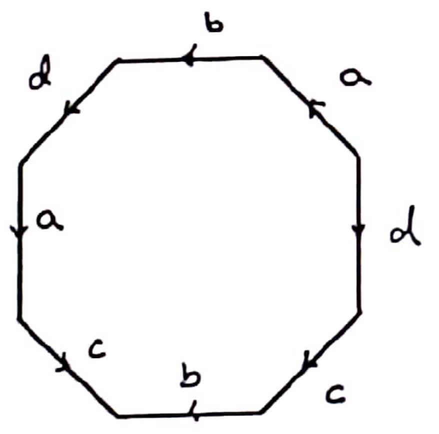

Figure 1: Planar representation for Exercise 5.

Exercise 6 (3 points). Consider the CW-complex X described as follows:

- It consists of a vertex p, an edge e, and a disc D.

- The endpoints of e are attached to p.
- The attaching map of D is given by $\partial D \cong \mathbb{S}^1 \ni z \to z^3 \in \mathbb{S}^1 \cong e$.

Compute the fundamental group of X.

Let $\tilde{X}$ be the universal cover of X.

- Describe the CW-structure that $\tilde{X}$ inherits from X. State explicitly how many cells it is made of and explain how they are attached pictorially.
- Describe the covering map $\tilde{X} \to X$.
- Prove that the $\tilde{X}$ you have described is indeed simply-connected.

Up to isomorphism, how many path-connected pointed covering spaces does (X, p) have?

Provide a list of all the path-connected 3-sheeted pointed covering spaces of $(\mathbb{S}^1, 1) \vee (X, p)$ (up to isomorphism). For each pointed covering space:

- Describe the CW-structure it inherits from the base (a good picture with appropriate labeling is enough).
- Compute its fundamental group and explain to which subgroup of the fundamental group of the base it corresponds to.
- Describe its deck transformations.

You must argue that your list is complete.