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In 2007/2008, the course WISB341 was given by dr. M. Crainic.

Topology and Geometry (WISB341)
April 15, 2008

Question 1

We consider the collection of subsets of $\mathbb{R}$:

$$\mathcal{B} := \{[a,b) : a, b \in \mathbb{R}, a \leq b\} \cup \{\mathbb{R}\}.$$

a) Show that $\mathcal{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. *(0.5 point)*

b) Let $\mathcal{T}$ be the smallest topology on $\mathbb{R}$ containing $\mathcal{B}$ Show that $\mathcal{T}$ is larger than the Euclidean topology $\mathcal{T}_{\text{eucl}}$. *(1 point)*

c) In the topological space $(\mathbb{R}, \mathcal{T})$, find the closure, the interior and the boundary of

$$A = (O, 1) \cup [2, 3].$$

(1.5 points)

d) Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{eucl}})$ are not homeomorphic. *(0.5 point)*

Question 2 *(1.5 points)*

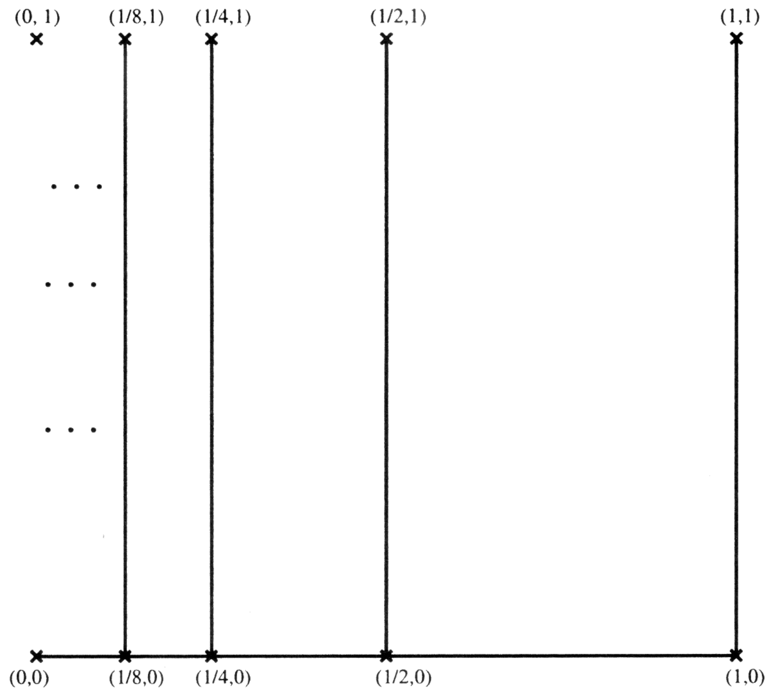
Show that the torus T contains a subspace C homeomorphic to a bouquet of two circles such that $T - C$ is homeomorphic to the open 2-disk. Similarly for the double torus and a bouquet of four circles.

Question 3 *(1.5 points)*

Consider the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ of reminders modulo n (with the adition modulo n), and the action of $\mathbb{Z}_n$ on the circle S^1 given by

$$k \bullet (\cos(t), \sin(t)) = (\cos(t + \frac{2k\pi}{n}), \sin(t + \frac{2k\pi}{n}))$$

(for $k \in \mathbb{Z}_n, (\cos(t), \sin(t)) \in S^1$). Show that the resulting quotient $S^1/\mathbb{Z}_n$ is homeomorphic to S^1. What can you say when $n = 2$?



Question 4

Consider the following subset of $\mathbb{R}^2$:

$$X = \bigcup_{n \geq 0 \text{integer}} \left\{ \frac{1}{2^n} \right\} \times [0,1] \cup [0,1] \times \{0\} \cup \{(0,1)\}$$

(see the picture), with the induced topology.
Explain which of the following properties are true for X.

a) it is Hausdorff. *(0.5 point)*

b) it is compact. *(1 point)*

c) it is locally compact. *(0.5 point)*

d) it is connected. *(0.5 point)*

e) it is path connected. *(0.5 point)*

Moreover, show that $X - \{(0,1)\}$ is locally compact and realize its one-point compactification as a subspace of $\mathbb{R}^2$. *(0.5 point)*