EXAM DIFFERENTIABLE MANIFOLDS, JANUARY 12 2009, 9:00–12:00

READ THIS FIRST. *Be sure to put your name and student number on every sheet you hand in. And if a solution continues an another sheet, or if you want part of the submitted solution sheets to be ignored by the grader, then clearly indicate so.*

You may do this exam either in Dutch or in English, but whichever language you choose, be clear and concise. Books or notes (and neighbors for that matter) are not to be consulted.

Maps and manifolds are assumed to be of class C^∞ unless stated otherwise.

A solution set will soon after the exam be linked at on the familiar Smooth Manifolds webpage at `http://www.math.uu.nl/people/looijeng`

(1) Given a map $f : M \to N$, prove that $F : M \to M \times N$, $F(p) = (p, f(p))$ is an embedding.

(2) Let M be a compact nonempty oriented m-manifold. Construct an m-form on M which is not exact.

(3) Let V be a vector field on a manifold M. We say that a differential form α on M is V*-invariant* if it is killed by the Lie derivative $\mathcal{L}_V$: $\mathcal{L}_V(\alpha) = 0$.
  (a) Prove that the exterior product of two V-invariant forms is V-invariant.
  (b) Suppose that V generates a flow $(H_t : M \to M)_{t\in\mathbb{R}}$. Prove that a differential form α on M is V-invariant if and only if $H_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$.
  (c) Describe the differential forms on $\mathbb{R}^m$ that are invariant under all the coordinate vector fields $\frac{\partial}{\partial x^i}$, $i = 1, \dots, m$.
  (d) Consider a product manifold $S^1 \times N$ (so N a manifold). A point of $S^1 \times N$ is denoted $(e^{i\tau}, x)$ with $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and $x \in M$. So $\frac{d}{d\tau}$ defines a vector field on this manifold. Determine the p-forms α on $S^1 \times N$ that are invariant under this vector field. (Do this in terms of the decomposition $\alpha = \alpha' + d\tau \wedge \alpha''$, with α' and α'' forms of degree p resp. $p-1$ that depend on τ.)

(4) We regard a 2-form on a manifold M as an antisymmetric function on pairs of vector fields. Let α be a 1-form on M.
  (a) Prove if α is exact, then $V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]) = 0$.
  (b) Prove that if α is exact and f is a function on M, then $V(f\alpha(W)) - W(f\alpha(V)) - f\alpha([V, W]) = df(V)\alpha(W) - df(W)\alpha(V)$.
  (c) Prove that for general α, $V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]) = d\alpha(V, W)$.

(5) We give S^2 its standard orientation. Denote by $\pi : S^2 \to P^2$ is the usual projection to the projective plane which identifies antipodal pairs. Prove that for every 2-form α on P^2, we have $\int_{S^2} \pi^*\alpha = 0$.