WORKED EXAM DIFFERENTIABLE MANIFOLDS, JANUARY 12 2009, 9:00–12:00

1. Given a map $f : M \to N$, prove that $F : M \to M \times N$, $F(p) = (p, f(p))$ is an embedding. According to Prop. 4.7 of Ch. 1 it suffices to prove that F is an immersion and a homeomorphism onto its image. Denote the image of f by Γ. The map $p \in M \mapsto (f, f(p)) \in \Gamma$ is continuous and bijective. Its inverse is the restriction to Γ of the projection $\pi_M : M \times N \to M$, and hence also continuous. So F maps M homeomorphically onto Γ. Since $\pi_M F$ is the identity, so is $D_p(\pi_M F) = D_{F(p)}\pi_M \circ D_p F$ and hence $D_p F$ is injective. So F is an immersion as well.

2. Let M be a compact nonempty oriented m-manifold. Construct an m-form on M which is not exact. Choose an oriented chart (U, κ) for M with the property that $\kappa(U)$ is the unit ball of $\mathbb{R}^m$. Let $f : \mathbb{R}^m \to [0, 1]$ be a smooth function that is not identically zero, but is zero outside the closed ball B centered at 0 of radius $\frac{1}{2}$. Then the integral $\int_{\mathbb{R}^m} f(x^1, \dots, x^m) dx^1 \dots dx^m$ converges and is a positive real number. Now let ω be the m-form on M characterized by the property that on U it is equal to $\kappa^*(f dx^1 \wedge \cdots \wedge dx^m)$ and is zero on $M - U$. (A formal proof that this ω is smooth goes as follows: B is compact, κ is a homeomorphism of U onto $\kappa(U)$ and so $\kappa^{-1}B$ is also compact. But M is Hausdorff and so this implies that $M - \kappa^{-1}B$ is open in M. Hence M is covered by its two open subsets U and $M - \kappa^{-1}B$. Since ω is smooth on either subset (it is identically zero on $M - \kappa^{-1}B$), ω is smooth.) Now by definition $\int_M \omega = \int_{\mathbb{R}^m} f(x^1, \dots, x^m) dx^1 \cdots d^m > 0$. Stokes' theorem then precludes this form to be exact, for if it were and $\omega = d\eta$, then that theorem says that $\int_M \omega = \int_{\partial M} \eta$ and this is zero since $\partial M = \emptyset$.

3. Let V be a vector field on a manifold M. We say that a differential form α on M is V-invariant *if it is killed by the Lie derivative $\mathcal{L}_V$: $\mathcal{L}_V(\alpha) = 0$.*
3a. Prove that the exterior product of two V-invariant forms is V-invariant. If α and β are differential forms on M, then we have a Leibniz rule asserting that $\mathcal{L}_V(\alpha \wedge \beta) = \mathcal{L}_V(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_V(\beta)$. So if $\mathcal{L}_V(\alpha) = 0$ and $\mathcal{L}_V(\beta) = 0$, $\mathcal{L}_V(\alpha \wedge \beta) = 0$.
3b. Suppose that V generates a flow $(H_t : M \to M)_{t \in \mathbb{R}}$. Prove that a differential form α on M is V-invariant if and only if $H_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$. Assigning to $t \in \mathbb{R}$ the form $H_t^*\alpha$ defines a function F from $\mathbb{R}$ to the vector space of differential forms. Now, $H_t^*\alpha$ is constant in t if and only if the derivative of F with respect to t is constant equal to zero. The derivative in $t = 0$ is $\mathcal{L}_V\alpha$. Since $\mathcal{L}_V$ commutes with H_t^*, its derivative in $t = t_0$ is $\mathcal{L}_V H_{t_0}^*\alpha = H_{t_0}^*(\mathcal{L}_V\alpha)$. The latter is zero if and only if $\mathcal{L}_V\alpha = 0$.
3c. Describe the differential forms on $\mathbb{R}^m$ that are invariant under all the coordinate vector fields $\frac{\partial}{\partial x^i}$, $i = 1, \dots, m$. The vector field $\frac{\partial}{\partial x^i}$ generates the flow that assigns to t the translation in t times the basis vector $e_i \in \mathbb{R}^m$. So invariance under all the coordinate vector fields amounts to translation invariance. This means: constant coefficients: such a k-form is a $\mathbb{R}$-linear combination of the forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq m$.
3d. Consider a product manifold $S^1 \times N$ (so N a manifold). A point of $S^1 \times N$ is denoted $(e^{i\tau}, x)$ with $\tau \in \mathbb{R}/(2\pi\mathbb{Z})$ and $x \in M$. So $\frac{d}{d\tau}$ defines a vector field on this manifold. Determine the p-forms α on $S^1 \times N$ that are invariant under this vector field. (Do this in terms of the decomposition $\alpha = \alpha' + d\tau \wedge \alpha''$, with α' and α'' forms of degree p resp. $p - 1$ that depend on τ.) We show that α is nvariant under $\frac{d}{d\tau}$ if and only if α' and α'' are independent of τ. This can be proved as in (3c) (so by means of an argument based on (3b)) or we can proceed as follows. Let us write d_N for the N-component of the exterior

derivative. Then

$$d\alpha = d_M\alpha' + d\tau \wedge \big(\frac{\partial \alpha'}{\partial \tau} - d_N\alpha''\big)$$

and so $\iota_{d/d\tau}d\alpha = \frac{\partial\alpha'}{\partial\tau} + d_N\alpha''$. We also see that

$$d\iota_{d/d\tau}\alpha = d\alpha'' = d_N\alpha'' + d\tau \wedge \frac{\partial \alpha''}{\partial \tau}.$$

It follows that the sum $\mathcal{L}_{d/d\tau}\alpha = \iota_{d/d\tau}d\alpha + d\iota_{d/d\tau}\alpha$ equals $\frac{\partial}{\partial\tau}\alpha' + d\tau \wedge \big(\frac{\partial}{\partial\tau}\alpha'\big)$. This is identically zero if and only if each term is, meaning that both α' and α'' are independent of τ.

4. We regard a 2-form on a manifold M as an antisymmetric function on pairs of vector fields. Let α be a 1-form on M.
4a. Prove if α is exact, then $V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]) = 0$. To say that α is exact means: $\alpha = df$ for some $f : M \to \mathbb{R}$. Then $\alpha(V) = d\alpha(V) = V(f)$ and similarly for W. So $V(\alpha(W)) - W(\alpha(V)) = VW(f) - WV(f) = [V, W](f) = \alpha([V, W])$.
4c. Prove that if α is exact and f is a function on M, then $V(f\alpha(W)) - W(f\alpha(V)) - f\alpha([V, W]) = df(V)\alpha(W) - df(W)\alpha(V)$. According to the Leibniz rule, we have $V(f\alpha(W)) = V(f)\alpha(W) + fV(\alpha(W)) = df(V)\alpha(W) + f\alpha(W)$ and likewise for $W(f\alpha(V))$. So

$$V(f\alpha(W)) - W(f\alpha(V)) = df(V)\alpha(W) + fV(\alpha(W)) - df(W)\alpha(V) - fW\alpha(V) =$$
$$df(V)\alpha(W) - df(W)\alpha(V) + f\alpha([V, W]) = (df \wedge \alpha)(V, W) + f\alpha([V, W])$$

4d. Prove that for general α, $V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]) = d\alpha(V, W)$. It is enough to verify this on a coordinate chart (U, κ). On that chart every 1-form is written as $f_1 d\kappa^1 + \cdots + f_m d\kappa^m$ for certain functions $f_1, \dots, f_m$ on U. According to (4c) the assertion holds for every term $f_i d\kappa^i$. So it holds for α.

5. We give S^2 its standard orientation. Denote by $\pi : S^2 \to P^2$ is the usual projection to the projective plane which identifies antipodal pairs. Prove that for every 2-form α on P^2, we have $\int_{S^2} \pi^*\alpha = 0$. We give two proofs, one lengthy, one short.

Long proof: Cover P^2 by finitely many charts (U_i, κ_i) with $\kappa(U_i)$ the unit ball and such that $\pi^{-1}U_i$ consists of two copies of U_i that are opposite with respect to the antipodal map. On only one of these copies the composite map $\kappa_i\pi$ is orientation preserving; denote that copy $\tilde{U}_i$. Then $\kappa_i\pi$ is on $-\tilde{U}_i$ orientation reversing and $\{\tilde{U}_i\}_i \cup \{-\tilde{U}_i\}_i$ covers S^2. We lift this covering to an oriented atlas by taking on $\tilde{U}_i$ the chart $\kappa_i\pi$ and on $-\tilde{U}_i$ the chart $\sigma\kappa_i\pi$, where $\sigma(x^1, x^2) = (-x^1, x^2)$. We now prove that $\int_{S^2} \pi^*\alpha = 0$ in case the support of α is contained in some U_i (this suffices for the general case then follows with the help of a partition of unity). For such α, let $f : \kappa(U_i) \to \mathbb{R}$ be such that $\alpha|U_i = \kappa_i^*(f dx^1 \wedge dx^2)$. Our definition prescribes that $\int_{\tilde{U}_i} \pi^*\alpha = \int_{\kappa(U_i)} f(x^1, x^2) dx^1 dx^2$ and that $\int_{-\tilde{U}_i} \pi^*\alpha = \int_{\sigma\kappa(U_i)} \sigma f dx^1 dx^2 = \int_{\kappa(U_i)} f(-x^1, x^2) dx^1 dx^2$. The last integral is (by the transformation formula) equal to $-\int_{\kappa(U_i)} f(x^1, x^2) dx^1 dx^2$. Hence $\int_{S^2} \pi^*\alpha = 0$.

Short proof: Let $\iota : S^2 \to S^2$, $\iota(x) = -x$, be the antipodal involution. Since ι reverses orientation we have that for any 2-form β on S^2: $\int_{S^2} \iota^*\beta = -\int_{S^2} \beta$. Now take $\beta := \pi^*\alpha$. Then $\iota^*\beta = \iota^*\pi^*\alpha = (\pi \circ \iota)^*\alpha = \pi^*\alpha = \beta$ and hence $\int_{S^2} \beta = 0$.