Differentiable manifolds – Exam 1

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten.

- Given a smooth map $f : M \longrightarrow N$, the *critical points of* f are the points $p \in M$ where $f_*|_p : T_pM \longrightarrow T_{f(p)}N$ is not surjective. The *critical values of* f are the points in N which are images of critical points.
- The *quaternions*, $\mathbb{H}$, are isomorphic to $\mathbb{R}^4$ as vector space and are endowed with a multiplication which makes $\mathbb{H}\backslash\{0\}$ into a group. This multiplication is $\mathbb{R}$-bilinear and is defined on a basis $\{1, i, j, k\}$ of $\mathbb{H}$ by
$$i^2 = j^2 = k^2 = ijk = -1.$$
- A *Lie group* is a differentiable manifold G endowed with the structure of a group and for which the maps
$$\begin{aligned} G \times G \longrightarrow G \qquad & (g, h) \mapsto g \cdot h \\ G \longrightarrow G \qquad & g \mapsto g^{-1} \end{aligned}$$
are smooth.

Questions

1) Show that the sphere
$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \subset \mathbb{R}^{n+1}$$
is a manifold.

2) Let M be the subset of $\mathbb{R}^3$ defined by the equation
$$M = \{(x_1, x_2, x_3) : x_1^3 + x_2^3 + x_3^3 = 1\}.$$

a) Show that M is a smooth submanifold of $\mathbb{R}^3$;

b) Define $\pi : M \longrightarrow \mathbb{R}$; $\pi(x_1, x_2, x_3) = x_1$. Find the critical points and critical values of π.

3) Let M be a compact manifold of dimension n and let $f : M \longrightarrow \mathbb{R}^n$ be smooth. Show that f has a critical point.

4a) Show that $\mathbb{H}\backslash\{0\}$ is a Lie group if endowed with quaternionic multiplication as group operation.
4b) Show that the 3-dimensional sphere $S^3 \subset \mathbb{H}\backslash\{0\}$ is also a Lie group with quaternionic multiplication as group operation.

5) Given a smooth map $\varphi : M \longrightarrow N$ it induces *pullback* maps on 0- and 1-forms, all of them denoted by φ^*, defined by

$$\begin{aligned} &\varphi^* : \Omega^0(N) \longrightarrow \Omega^0(M) && \varphi^* f = f \circ \varphi; \\ &\varphi^* : \Omega^1(N) \longrightarrow \Omega^1(M) && \alpha \mapsto \varphi^* \alpha; \\ & && \varphi^* \alpha|_p(X) = \alpha|_{\varphi(p)}(\varphi_* X) \qquad \forall X \in T_p M. \end{aligned}$$

Show that if $f \in \Omega^0(N)$, then $\varphi^* df = d(\varphi^* f)$.