Differentiable manifolds – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Let M be the subset of $\mathbb{R}^3$ defined by the equation

$$M = \{(x_1, x_2, x_3) : x_1x_2^2 + x_2x_3^2 + x_3x_1^2 = 1\}.$$

a) Show that M is a smooth submanifold of $\mathbb{R}^3$;

b) Define $\pi : M \longrightarrow \mathbb{R}$; $\pi(x_1, x_2, x_3) = x_1$. Find the critical points and critical values of π.

Solution.

a) M is the zero-level set of the function $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$F(x_1, x_2, x_3) = x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 1$$

hence to prove that M is a manifold it is enough to prove that 0 is a regular value, that is, prove that if $F(x) = 0$ then $F_*|x : T_x\mathbb{R}^3 \longrightarrow T_0\mathbb{R}$ is onto. For real valued functions, F_* is just the differential dF and since $T_0\mathbb{R}$ is a one dimensional vector space, dF is surjective whenever it is not zero. So, to prove that $0 \in \mathbb{R}$ s a regular value we must show that the equations $F(x) = 0$ and $dF(x) = 0$ do not have a solution. Since

$$dF(x_1, x_2, x_3) = (x_2^2 + 2x_3x_1)dx_1 + (x_3^2 + 2x_1x_2)dx_2 + (x_1^2 + 2x_2x_3)dx_3$$

we can spell out the conditions $F(x) = 0$ and $dF(x) = 0$:

$$\begin{cases} x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 1 = 0 \\ x_2^2 + 2x_3x_1 = 0 \\ x_3^2 + 2x_1x_2 = 0 \\ x_1^2 + 2x_2x_3 = 0 \end{cases}$$

First we observe that there is no solution with $x_1x_2x_3 = 0$. Indeed, say $x_1 = 0$, second and third equations give $x_2 = x_3 = 0$, but $(0, 0, 0)$ is not a solution to the first equation. If we divide the last two equations written as $x_3^2 = -2x_1x_2$ and $x_1^2 = -2x_2x_3^2$ we get

$$(x_3/x_1)^2 = (x_1/x_3) \Rightarrow x_3 = x_1.$$

By symmetry of the last three equations, we also get $x_2 = x_1$ and hence a solution to the last three equations should satisfy $3x_1^2 = 0$ hence $x_1 = 0$, but was saw that this can not be a solution to the system, so 0 is a regular value.

b) Similarly to the previous argument, the critical points x of $\pi|_M$ are those for which $d\pi_x : T_xM \longrightarrow T_{\pi(x)}\mathbb{R}$ is the zero map, that is $\ker(d\pi_x) = T_xM = \ker dF$, hence the points where $d\pi_x = \lambda dF_x$ are the critical points of π. Since $d\pi = dx_1$, we want to find the points in M where the coefficients of dF corresponding to dx_2 and dx_3 vanish, i.e., we want to solve

$$\begin{cases} x_1x_2^2 + x_2x_3^2 + x_3x_1^2 - 1 = 0 \\ x_3^2 + 2x_1x_2 = 0 \\ x_1^2 + 2x_2x_3 = 0 \end{cases}$$

Again, if, a solution has, say, $x_1 = 0$, then the last two equations give either x_2 or x_3 must vanish and hence the first equation can not hold. Similarly, if $x_2 =$, the last two equations imply $x_1 = x_3 = 0$ and the first equation does not hold. Following the same computations we did in a), we can rearrange the last two equations and divide them by each other to obtain

$$(x_3/x_1)^2 = x_1/x_3 \Rightarrow x_1 = x_3.$$

Then the last equation furnishes $x_2 = -x_1/2$ and the first gives

$$3x_1^3 = 1 \Rightarrow x_1 = \sqrt[3]{4/3},$$

So the point $(\sqrt[3]{4/3}, -\sqrt[3]{1/6}, \sqrt[3]{4/3})$ is the only critical point of π on M and the corresponding critical value is $\sqrt[3]{4/3}$.

2) Show that a smooth map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ can not be injective.

Solution.

Firstly we observe that if dF is identically zero, then F is constant and hence not an injection.

If $dF_p \neq 0$ for some $p \in \mathbb{R}^2$, then one of the several corollaries of the inverse function theorem states that we can find coordinates y in a nhood of $f(p)$ and coordinates (x_1, x_2) in a nhood of p for which $f(x_1, x_2) = x_1$. Hence f is not injective.

3) Let $M \stackrel{\varphi}{\longrightarrow} N$ be an embedded submanifold for which $\varphi(M)$ is a closed subset of N. Show that if $X \in \mathfrak{X}(M)$, then there exists a vector field $\tilde{X} \in \mathfrak{X}(N)$ which is φ-related to X. Such $\tilde{X}$ is normally called an *extension* of X to N. Given $X, Y \in \mathfrak{X}(M)$, let $\tilde{X}, \tilde{Y}$ be extensions of X and Y to N. Show that for $p \in \varphi(M)$, $[\tilde{X}, \tilde{Y}](p)$ is tangent to $\varphi(M)$ and depends only on X and Y and not on the particular extensions $\tilde{X}$ and $\tilde{Y}$ chosen.

Solution. From one of the several corollaries of the inverse function theorem, we know that for every $p \in M$ there is a coordinate system X in a nhood U of p and a coordinate system Y in a nhood V of $\varphi(p)$ such that the local expression for φ, $\tilde{\varphi} = Y \circ \varphi \circ X^{-1}$ is simply

$$\varphi(x) = (x, 0).$$

Since φ is an embedding, we can further assume that $V \cap \varphi(M) = \varphi(U)$.

Now we define $\tilde{X}$ in the nhood V (in the coordinates above) by

$$\tilde{X}(y_1, y_2) = X(y_1), \qquad y_1 \in \mathbb{R}^m,\ y_2 \in \mathbb{R}^{n-m},$$

that is, in these coordinates $\tilde{X}$ is independent of the last $n-m$ variables and, for $q=\varphi(p)$, $\tilde{X}(q)=\varphi_*|_p X(p)$.

This procedure can now be carried out in nhoods of all points of M to obtain an open cover U_α of M and corresponding extensions $\tilde{X}_\alpha$ of $\varphi_* X$ to V_α., so for $p \in U_\alpha$ $\varphi_p * X(p) = \tilde{X}(\varphi(p))$. By second countability we can find a locally finite and countable refinement of U_α which we still denote by U_α and we still denote the corresponding vector fields $X_\alpha \in \mathfrak{X}(V_\alpha)$. Since $\varphi(M)$ is closed, $N\backslash\varphi(M)$ is open and the collection formed by (V_α) and the open set $N\backslash\varphi(M)$ is a locally finite cover of N, hence we can find a partition of unity (ξ_α) subordinated to this cover with same index set.

Now define $\tilde{X} = \sum \xi_\alpha X_\alpha$. Then, for $p \in M$ and $q=\varphi(p)$ we have

$$\tilde{X}(q) = \sum_\alpha \xi_\alpha(q)\tilde{X}_\alpha(q) = \sum_\alpha \xi_\alpha(q)\varphi_*|_p X(p) = \varphi_*|_p X(p).$$

Therefore $\tilde{X}$ is φ-related to X.

Now given vector fields $X, Y \in \mathfrak{X}(M)$ and extensions $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$. Then, by the above these vector fields are φ-related and hence so is their Lie bracket:

$$\varphi_*|_p([X,Y]) = [\tilde{X},\tilde{Y}](q).$$

Since the quantity in the left hand side of the expression above is independent of the choices of extensions, so is the quantity on the right hand side, i.e., $[\tilde{X},\tilde{Y}](q)$depends only on X, Y but not on the extensions chosen.

4) Show that $\mathbb{C}\backslash\{0\}$ with complex multiplication is a Lie group. Show that S^1, the set of complex numbers of norm 1, is also a Lie group.

First we will show that $C^* = \mathbb{C}\backslash\{0\}$ is a Lie group, i.e., that multiplication and inversion are smooth. Notice that $\mathbb{C}^* = \mathbb{R}^2\backslash\{0\}$, $x+iy \mapsto (x,y)$ is covered by a single chart, so to check smoothness we can simply check it in this chart.

Multiplication is given by

$$(x+iy, u+iv) \mapsto xu - yv + i(xv+yu),$$

or, in coordinates

$$((x,y),(u,v)) \mapsto (xu-yv, xv+yu)$$

and we see that the map is polynomial on the coordinates (x,y) and (u,v), hence smooth.

Inversion is given by

$$z \mapsto \overline{z}/\|z\|^2$$

in coordinates, this is

$$(x,y) \mapsto \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right),$$

which is clearly smooth on $\mathbb{R}^2\backslash\{0\}$.

Now we check that S^1 is a Lie group. Since multiplication and inversion are smooth in $\mathbb{C}^*$, their restriction to S^1 is also smooth, i.e.

$$S^1 \times S^1 \longrightarrow \mathbb{C}^* \qquad (z_1,z_2) \longrightarrow z_1 z_2$$

$$S^1 \longrightarrow \mathbb{C}^* \qquad z \longrightarrow z^{-1}$$

are smooth.

Since S^1 is a subgroup of $\mathbb{C}^*$, as has been checked in the group theory course, the image of maps above is S^1 and since S^1 is an embedded submanifold of $\mathbb{C}^*$ this means that the maps

$$S^1 \times S^1 \longrightarrow S^1 \subset \mathbb{C}^* \qquad (z_1, z_2) \longrightarrow z_1 z_2$$

$$S^1 \longrightarrow S^1 \subset \mathbb{C}^* \qquad z \longrightarrow z^{-1}$$

are also smooth (c.f. Warner Theorem 1.32), hence S^1 is a Lie (sub)group.

5) Let $(U_\alpha : \alpha \in A)$ be an open cover of a manifold M and let $f_\alpha : U_\alpha \longrightarrow \mathbb{R}$ be a family of smooth functions such that on $U_\alpha \cap U_\beta$, $f_\alpha - f_\beta$ is constant, for all $\alpha, \beta \in A$. Show that if we define a 1-form ξ on M by declaring that, on U_α, $\xi = df_\alpha$, then ξ is a globally defined 1-form.

Solution.

Define $\xi_\alpha \in \Omega^1(U_\alpha)$ by $\xi_\alpha = df_\alpha$. Then if $x \in U_\alpha \cap U_\beta$ we have

$$\xi_\alpha - \xi_\beta = df_\alpha - df_\beta = d(f_\alpha - f_\beta) = 0.$$

Hence the form ξ defined to be equal to ξ_α in U_α is well defined (its value at a point x does not depend on which representative ξ_α was used to define it). Since f_α is smooth, so is ξ.