Differentiable manifolds – Exam 3

Notes:

1. **Write your name and student number** ** *clearly* ** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** The complex projective n-space, denoted by $\mathbb{C}P^n$, is the set of all 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$ with the quotient topology inherited from the natural projections $\pi : \mathbb{C}^{n+1}\backslash\{0\} \longrightarrow \mathbb{C}P^n$. Show that $\mathbb{C}P^n$ is a compact $2n$-dimensional topological manifold and show how to give it a smooth structure analogous to the one given in lectures to $\mathbb{R}P^n$.

Exercise 2 (2.0 pt)**.** Let M be a manifold and $\varphi : S \longrightarrow M$ be an injective immersion (i.e., a submanifold). Show that φ is an embedding **if and only if** every smooth function $f : S \longrightarrow \mathbb{R}$ has an extension to a neighborhood U of $\varphi(S)$. That is, for every $f : S \longrightarrow \mathbb{R}$ smooth, there is an open $U \subset M$ with $\varphi(S) \subset U$ and a smooth map $g : U \longrightarrow \mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "g"] & \mathbb{R} \\ S \arrow[u, "\varphi"] \arrow[ur, "f"'] & \end{tikzcd}$$

Exercise 3 (2.0 pt)**.** Given a manifold M, the space of sections of the bundle $TM \oplus T^*M$ is endowed with the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\eta(X) + \xi(Y))$$

and a bracket (the *Courant bracket*):

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - i_Y d\xi, \qquad X, Y \in \Gamma(TM);\ \xi, \eta \in \Gamma(T^*M).$$

1. Given a 2-form $B \in \Omega^2(M)$, let L be the subbundle of $TM \oplus T^*M$ given by
$$L = \{X - i_X B : X \in TM\}.$$
Letting $\Gamma(L)$ be the space of sections of L, show that
$$[\alpha, \beta] \subset \Gamma(L), \qquad \text{for all } \alpha, \beta \in \Gamma(L)$$
if and only if B is closed.

2. Show that for $X, Y, Z \in \Gamma(TM)$ and $\xi, \eta, \mu \in \Gamma(T^*M)$ we have
$$\mathcal{L}_X \langle Y + \eta, Z + \mu \rangle = \langle [X + \xi, Y + \eta], Z + \mu \rangle + \langle Y + \eta, [X + \xi, Z + \mu] \rangle.$$

Exercise 4 (2.0 pt)**.** Let ρ be the following form defined on $\mathbb{R}^3 \backslash \{0\}$
$$\rho = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

1. Compute $d\rho$;
2. Compute the integral of ρ over the 2-sphere of radius 1 in $\mathbb{R}^3$ centered at the origin;
3. Compute the integral of ρ over the 2-torus in $\mathbb{R}^3$ parametrized by
$$S^1 \times S^1 \longrightarrow \mathbb{R}^3 \qquad (\theta, \varphi) \mapsto ((\cos\theta + 4)\cos\varphi, (\cos\theta + 4)\sin\varphi, \sin\theta)$$
4. Is $H^2(\mathbb{R}^3 \backslash \{0\})$ trivial or not?

Exercise 5 (2.0 pt)**.** Compute the degree one de Rham cohomology of

1. $\mathbb{R}^2 \backslash \{0\}$;
2. $\mathbb{R}^n \backslash \{0\}$ for $n > 2$.