Differentiable manifolds – Exam 2

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.

Some useful definitions and results

- **Definition**. A *star shaped domain* of $\mathbb{R}^n$ is an open set $U \subset \mathbb{R}^n$ such that there is $p \in U$ with the property that if $q \in U$, then all the points in the segment connecting p and q are also in U, that is, there is p such that
$$(1-t)p + tq \in U; \text{ for all } q \in U \text{ and all } t \in [0,1].$$
The Poincaré Lemma states

  Theorem 1 (Poincaré Lemma)**.** *If U is (diffeomorphic to) a star shaped domain of $\mathbb{R}^n$ then*
$$H^k(U) = \{0\} \qquad \text{for } k > 0.$$

Questions

Exercise 1 (2.5 pt)**.** Suppose M is a smooth n-dimensional manifold and $S \subset M$ is an embedded compact submanifold. Suppose further that there is a smooth vector field X defined on a neighborhood of S and which is nowhere tangent to S. Show that there exists $\varepsilon > 0$ such that the flow of X restricts to a smooth embedding
$$\Phi : (-\varepsilon, \varepsilon) \times S \longrightarrow M; \qquad \Phi(t,p) = e^{tX}(p).$$

Exercise 2 (2.5 pt)**.** Let $\alpha \in \Omega^2(\mathbb{R}^3)$ be given by
$$\alpha = x dy dz$$
Compute the integral of α over

a) The unit sphere;

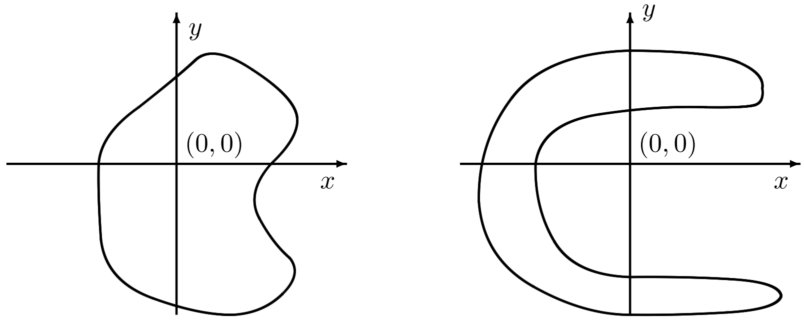
b) The torus parametrized by
$$S^1 \times S^1 \longrightarrow \mathbb{R}^3; \qquad (\theta, \varphi) \mapsto ((\cos\theta + 2)\cos\varphi, (\cos\theta + 2)\sin\varphi, \sin\theta).$$

Exercise 3 (2.5 pt)**.** Consider the form $\rho \in \Omega^1(\mathbb{R}^2 \backslash \{0\})$
$$\rho = \frac{x dy - y dx}{x^2 + y^2}$$

a) Show that $d\rho = 0$;

b) Compute the integral of ρ over the circle of radius r centered at the origin.

c) Compute the integral of ρ over the paths drawn below in $\mathbb{R}^2$, both traced counterclockwise.



d) Does ρ represent a nontrivial cohomology class in $\mathbb{R}^2 \backslash \{0\}$? Does ρ represent a nontrivial class in

$$\mathbb{R}^2 \backslash \{(x, 0) : x \geq 0\}?$$

Exercise 4 (2.5 pt)**.** Let $\Omega^\bullet(M)$ be the graded vector space of differential forms on M and $\mathfrak{L}(\Omega^\bullet(M))$ be the graded space of linear endomorphisms of $\Omega^\bullet(M)$ endowed with the graded commutator as a bracket

$$\{\cdot, \cdot\} : \mathfrak{L}^k(\Omega^\bullet(M)) \times \mathfrak{L}^l(\Omega^\bullet(M)) \longrightarrow \mathfrak{L}^{k+l}(\Omega^\bullet(M)),$$

$$\{A, B\} = AB + (-1)^{kl+1} BA.$$

Recall that interior product by a vector field $X \in \mathfrak{X}(M)$ is a map of degree -1, wedge product by a 1-form ξ and the exterior derivative are maps of degree $+1$. To simplify notation we denote $\xi \wedge \varphi$ simply by $\xi \cdot \varphi$ and $\iota_X \varphi$ by $X \cdot \varphi$. The point of this exercise is to define a natural bracket on $\Gamma(TM \oplus T^*M)$ and determine some of its basic properties.

a) Show that for all $\varphi \in \Omega^\bullet(M)$, $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$,

$$\{\{X + \xi, d\}, Y + \eta\} \cdot \varphi = ([X, Y] + (\mathcal{L}_X \eta) - (\iota_Y d\xi)) \cdot \varphi.$$

Define a bracket $[\![\cdot, \cdot]\!] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \longrightarrow \Gamma(TM \oplus T^*M)$ by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi$$

for $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

b) Show that for $f \in C^\infty(M)$

$$[\![X + \xi, f(Y + \eta)]\!] = f[\![X + \xi, Y + \eta]\!] + (\mathcal{L}_X f)(Y + \eta).$$

c) Show that

$$[\![X + \xi, X + \xi]\!] = d(\xi(X)).$$

d) Show that if $\omega \in \Omega^2(M)$ is a closed form then

$$[\![X + \xi + \iota_X \omega, Y + \eta + \iota_Y \omega]\!] = [\![X + \xi, Y + \eta]\!] + \iota_{[X,Y]}\omega.$$