Differentiable manifolds – Exam 1

Notes:

1. **Write your name and student number *****clearly******* on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** Let M be a nonempty smooth compact manifold. Show that

1. there is no submersion $f : M \longrightarrow \mathbb{R}$.
2. there is no submersion $f : M \longrightarrow \mathbb{R}^k$ for any $k > 0$.

Exercise 2. (2.0 pt) Let M be a manifold and let $f, g : M \longrightarrow \mathbb{R}$ be smooth real functions. Let y be a regular value of g and let $P = g^{-1}(y)$. Show that a point $x \in P$ is a critical point for the restriction

$$f|_P : P \longrightarrow \mathbb{R}$$

if and only if there is a real number λ such that $df|_x = \lambda dg|_x$.

Exercise 3. (2.0 pt) Let $\varphi : N \longrightarrow M$ be a submanifold and let $\gamma : (0, 1) \longrightarrow M$ be a smooth path which lies in the image of φ, i.e., $\gamma((0, 1)) \subset \varphi(N)$.

1. Show that if φ is an embedding, then $\frac{d\gamma}{dt}|_t \in \varphi_*(T_{\varphi^{-1}(\gamma(t))}N)$;
2. Give an example of a submanifold which is not an embedding for which the conclusion above fails.

Exercise 4 (2.0 pt)**.** Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on TM, i.e.,

- for every $p \in M$
$$\langle \cdot, \cdot \rangle : T_pM \times T_pM \longrightarrow \mathbb{R}$$
is linear on both entries,
- for every $p \in M$ and every $X \in T_pM \backslash \{0\}$,
$$\langle X, X \rangle > 0$$

- For any $X, Y \in \mathcal{X}(M)$ smooth vector fields, $\langle X, Y\rangle$ is a smooth real function.

1. Given $f : M \longrightarrow \mathbb{R}$, show that there is a smooth vector field $\nabla f : M \longrightarrow TM$ such that for all $X \in \mathcal{X}(M)$, $\langle \nabla f, X\rangle = df(X)$.
2. Show that if $y \in \mathbb{R}$ is a regular value of f then ∇f is orthogonal to the tangent space of the submanifold $f^{-1}(y)$, i.e., if $f(p) = y$ and X is tangent to $f^{-1}(y)$ at p, then $\langle \nabla f, X\rangle = 0$.
3. Let $p \in M$ be a point such that $\nabla f|_p \neq 0$ and let

$$X = \nabla f / \langle \nabla f, \nabla f\rangle^{1/2}\Big|_p \in T_pM.$$

   Show that for any $Y \in T_pM$ with $\langle Y, Y\rangle = 1$

$$\mathcal{L}_Y f|_p \leq \mathcal{L}_X f|_p.$$

Exercise 5 (2.0 pt)**.** Prove or give a counterexample to the following claim: "If $E \longrightarrow M$ is a nontrivial bundle, then $E \oplus E \longrightarrow M$ is also nontrivial."