Differentiable manifolds – Exam 3

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.

Some useful definitions and results

- **Definition**. A *star shaped domain* of $\mathbb{R}^n$ is an open set $U \subset \mathbb{R}^n$ such that there is $p \in U$ with the property that if $q \in U$, then all the points in the segment connecting p and q are also in U, that is, there is p such that
$$(1-t)p + tq \in U; \text{ for all } q \in U \text{ and all } t \in [0,1].$$
The Poincaré Lemma in full generality states

  Theorem 1 (Poincaré Lemma)**.** *If U is (diffeomorphic to) a star shaped domain of $\mathbb{R}^n$ then*
$$H^k(U) = \{0\} \qquad \text{for } k > 0.$$

- **Definition**. An open cover $\mathcal{U}$ of a manifold M is *fine* if any finite intersection of elements of $\mathcal{U}$ is either empty or (diffeomorphic to) a disc.

  With this definition, we have

  Theorem 2 (Čech to de Rham)**.** *The Čech cohomology with real coefficients of any fine cover of M is isomorphic to the de Rham cohomology of M.*

Questions

Exercise 1. (1 pt) Let V be a vector space. Show that if $\dim(V) = 3$, then every homogeneous element of degree greater than zero in $\wedge^\bullet V$ is decomposable, i.e., can be written as a product of degree one elements.

Exercise 2. (1.5 pt) Let $\pi : M \longrightarrow N$ be a submersion, i.e., π is a surjection and $\pi_* : T_pM \longrightarrow T_{\pi(p)}N$ is a surjection for all $p \in M$. We call a vector $v \in T_pM$ *vertical* if $\pi_*(v) = 0$. Show that if for $\alpha \in \Omega^k(M)$ the following hold

1. $\iota_v \alpha = 0$ for all vertical vectors;
2. $\mathcal{L}_v \alpha = 0$ for all vertical vector fields and
3. $\pi^{-1}(p)$ is connected for all $p \in N$,

then there is $\beta \in \Omega^k(N)$ such that $\alpha = \pi^*\beta$.

Exercise 3.

1. (1 pt) Show that $\mathbb{RP}^n$, the set of complex lines through the origin in $\mathbb{R}^{n+1}$, can be given the structure of a compact manifold.

2. (0.5 pt) Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a homogeneous polynomial of degree m in three variables, i.e.,

$$p(X_0, X_1, X_2) = \sum_{i+j+k=m} a_{ijk} X_0^i X_1^j X_2^k.$$

Let $\Sigma \subset \mathbb{R}P^2$ be the set defined by the zeros of p, i.e.,

$$\Sigma = \{[X_0, X_1, X_2] \in \mathbb{R}P^2 : p(X_0, X_1, X_2) = 0\}.$$

Show that Σ is indeed a well defined subset of $\mathbb{R}P^2$, i.e., if two points (different from 0) are in the same line through the origin, then either both are zeros of p or neither is a zero of p.

3. (1 pt) Show that if the system

$$\begin{cases} p(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_0}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_1}(X_0, X_1, X_2) & = 0, \\ \frac{\partial p}{\partial X_2}(X_0, X_1, X_2) & = 0, \end{cases}$$

has no solutions other than $(0, 0, 0)$, then Σ is an embedded submanifold of $\mathbb{R}P^2$.

Exercise 4.

1. (1 pt) Using the results of the previous exercise or otherwise, prove that the zeros of the polynomial

$$p : \mathbb{R}^3 \longrightarrow \mathbb{R}, \qquad p(X_0, X_1, X_2) = X_0^3 - X_1(X_1 - X_2)(X_1 - 2X_2)$$

define a smooth submanifold $\Sigma \subset \mathbb{R}P^2$. (Hint: use that $p(0, x, 1)$ and $p(0, 1, x)$ have simple roots).

2. (1.5 pt)Let $\pi : \Sigma \longrightarrow \mathbb{R}P^1$ be defined by

$$\pi([X_0, X_1, X_2]) = [X_1, X_2].$$

Find the critical points of π.

Exercise 5. Consider the following 2-form defined in $\mathbb{R}^4$:

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4.$$

1. (0.5 pt) Compute $d\omega$;

2. (1 pt) Consider the following map

$$\varphi : S^2 \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^4, \qquad \varphi(a, b, c) = (a, b, 2ac, 2bc)$$

Compute

$$\int_{S^2} \varphi^* \omega.$$

3. (1 pt) Consider the following map

$$\varphi : S^1 \times S^1 \longrightarrow \mathbb{R}^4, \qquad \varphi(\theta_1, \theta_2) = (\sin(\theta_1)\cos(\theta_2), \sin(\theta_1)\sin(\theta_2), \cos(\theta_1), \sin(\theta_2))$$

Compute

$$\int_{S^1 \times S^1} \varphi^* \omega.$$