# Differentiable manifolds – Exam 2

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.

## Some useful definitions and results

- **Definition**. A *star shaped domain* of $\mathbb{R}^n$ is an open set $U \subset \mathbb{R}^n$ such that there is $p \in U$ with the property that if $q \in U$, then all the points in the segment connecting $p$ and $q$ are also in $U$, that is, there is $p$ such that
$$(1-t)p + tq \in U; \text{ for all } q \in U \text{ and all } t \in [0,1].$$
The Poincaré Lemma in full generality states

  **Theorem 1** (Poincaré Lemma)**.** *If $U$ is (diffeomorphic to) a star shaped domain of $\mathbb{R}^n$ then*
$$H^k(U) = \{0\} \qquad \text{for } k > 0.$$

- **Definition**. An open cover $\mathcal{U}$ of a manifold $M$ is *fine* if any finite intersection of elements of $\mathcal{U}$ is either empty or (diffeomorphic to) a disc.

  With this definition, we have

  **Theorem 2** (Čech to de Rham)**.** *The Čech cohomology with real coefficients of any fine cover of $M$ is isomorphic to the de Rham cohomology of $M$.*

## Questions

**Exercise 1.** Let $V$ be a vector space.

a) (0.5 pt) Let $\xi \in \wedge V^*$ be an odd form. Show that $\xi \wedge \xi = 0$.

b) (0.5 pt) Give an example of an even form $\xi$ such that $\xi \wedge \xi \neq 0$.

b) (1 pt) Let $\xi \in V^*\backslash\{0\}$ and $\eta \in \wedge^k V^*$. Show that $\xi \wedge \eta = 0$ if and only if there is $\zeta \in \wedge^{k-1}V^*$ such that $\eta = \xi \wedge \zeta$.

**Exercise 2.** Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be given by
$$\alpha = xdy$$
Compute the integral of $\alpha$ over

a) (1 pt) The unit circle centered at the origin parametrized counterclockwise;

b) (1 pt) The boundary of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ parametrized counterclockwise.

**Exercise 3.** (2 pt) Consider the form $\rho \in \Omega^2(\mathbb{R}^3 \backslash \{0\})$

$$\rho = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

a) Show that $d\rho = 0$;

b) Compute the integral of $\rho$ over the 2-sphere of radius 2 in $\mathbb{R}^3$ centered at $(0, 0, 1)$.

c) Compute the integral of $\rho$ over the 2-sphere of radius 2 in $\mathbb{R}^3$ centered at $(0, 0, 3)$.

d) Does $\rho$ represent a nontrivial cohomology class in $\mathbb{R}^3 \backslash \{0\}$? Does $\rho$ represent a nontrivial class in

$$\mathbb{R}^3 \backslash \{(0, 0, x) : x \geq 0\}?$$

**Exercise 4** (2 pt)**.** Let $M$ be a manifold. Endow the bundle $TM \oplus T^*M$ with the bracket

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi,$$

where $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$ and $[X, Y]$ is the Lie bracket of $X$ and $Y$.

Given $\omega \in \Omega^2(M)$ and $X \in \mathfrak{X}(M)$ interior product gives $\iota_X \omega \in \Omega^1(M)$. Show that the following are equivalent:

1. For all $X, Y \in \mathfrak{X}(M)$ there is $Z \in \mathfrak{X}(M)$ such that

$$[\![X + \iota_X \omega, Y + \iota_Y \omega]\!] = Z + \iota_Z \omega.$$

2. $d\omega = 0$.

**Exercise 5.** Compute the dimension of the degree 1 de Rham cohomology of

a) (1 pt) $\mathbb{R}^2 \backslash \{(0, 0)\}$.

b) (1 pt) $\mathbb{R}^2 \backslash \{(0, 0), (1, 0), \cdots, (n, 0)\}$, where $n$ is some positive integer.