Differentiable manifolds – Exam 2

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **allowed** to consult any text book and class notes but **not allowed** to consult colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten.

- An n dimensional complex manifold is a manifold whose charts take values in $\mathbb{C}^n$ and for which the change of coordinates are holomorphic maps.
- A volume form on a manifold M^n is a nowhere vanishing n-form.

Questions

1) Show that $\mathbb{CP}^n$, the set of complex lines through the origin in $\mathbb{C}^{n+1}$, can be given the structure of a complex manifold.

2) Given a manifold M, the space of sections of the bundle $TM \oplus T^*M$ is endowed with the natural pairing

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}(\eta(X)+\xi(Y))$$

and a bracket (the *Courant bracket*):

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi, \qquad X,Y\in\Gamma(TM);\ \xi,\eta\in\Gamma(T^*M).$$

a) Given a 2-form $B\in\Omega^2(M)$, let L be the subbundle of $TM\oplus T^*M$ given by

$$L = \{X - i_X B : X\in TM\}.$$

Show that L is involutive with respect to the Courant bracket if and only if B is closed.

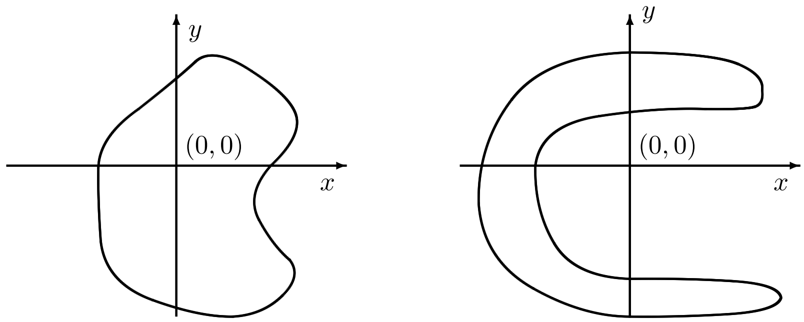
b) Show that for $X,Y,Z\in\Gamma(TM)$ and $\xi,\eta,\mu\in\Gamma(T^*M)$ we have

$$\mathcal{L}_X\langle Y+\eta, Z+\mu\rangle = \langle [\![X+\xi, Y+\eta]\!], Z+\mu\rangle + \langle Y+\eta, [\![X+\xi, Z+\mu]\!]\rangle.$$

3) Compute the integral of the 1-form

$$\theta = \frac{xdy - ydx}{x^2 + y^2}.$$

along the paths drawn below traced counterclockwise.



4) For $i \in \mathbb{N}$, let $p_i = (i, 0) \in \mathbb{R}^2$. For $n \in \mathbb{N}$, compute the degree one de Rham cohomology of $\mathbb{R}^2 \backslash \{p_1, \cdots, p_n\}$.

5) Show that every manifold admits a Riemannian metric.