Solutions to the final exam

Questions

Exercise 1(20 pt) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x, y, z) := x^2 + y^2 - z^2$.

a) For which $c \in \mathbb{R}$ is $M_c := F^{-1}(c)$ a smooth submanifold of $\mathbb{R}^3$? Give a sketch of M_c for all $c \in \mathbb{R}$.

Solution: The derivative of F is given by $d_{(x,y,z)}F = (2x, 2y, -2z) : \mathbb{R}^3 \to \mathbb{R}$ and is surjective for all $(x, y, z) \neq 0$. So $(0, 0, 0) \in F^{-1}(0)$ is the only point where F is not a submersion, showing that $M_c = F^{-1}(c)$ is a smooth submanifold of $\mathbb{R}^3$ for all $c \neq 0$. (Sketch: see last page of this file)

M_0 is not a smooth submanifold, in fact it is not even a topological manifold! It is given by a cone, and a neighborhood of the tip of the cone can not be homeomorphic to a disc, for it becomes disconnected if you remove the tip itself.

b) Show that M_1 is diffeomorphic to $S^1 \times \mathbb{R}$ and that M_{-1} is diffeomorphic to $\mathbb{R}^2 \coprod \mathbb{R}^2$.

Solution: $M_1 = \{(x, y, z)|x^2 + y^2 = 1 + z^2\}$. Since $1 + z^2$ is always positive, this is a union of circles as z varies. Explicitly, define $\varphi : S^1 \times \mathbb{R} \to M_1$ by $\varphi(a, b, t) := (\sqrt{1 + t^2}a, \sqrt{1 + t^2}b, t)$ where $t \in \mathbb{R}$ and $(a, b) \in S^1 \subset \mathbb{R}^2$ (seen as the unit circle). This is a smooth map because it is the restriction of the smooth map from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^3$ given by the same formula. Define $\psi : M_{-1} \to S^1 \times \mathbb{R}$ by $\psi(x, y, z) := (\frac{1}{\sqrt{1+z^2}}x, \frac{1}{\sqrt{1+z^2}}y, z)$. This is again a smooth map and it is inverse to φ.

Next consider $M_{-1} = \{(x, y, z) \in \mathbb{R}^3|z^2 = x^2 + y^2 + 1\} = M^+_{-1} \coprod M^-_{-1}$ where $M^{\pm}_{-1} := \{(x, y, z) \in \mathbb{R}^3|z = \pm\sqrt{x^2 + y^2 + 1}\}$. Both $M^{\pm}_{-1}$ are the graph of a smooth function on $\mathbb{R}^2$, hence are diffeomorphic to $\mathbb{R}^2$. (The diffeomorphism itself can be taken to be the projection $(x, y, z) \mapsto (x, y)$, with inverse $(x, y) \mapsto (x, y, \pm\sqrt{x^2 + y^2 + 1})$.)

Exercise 2(20 pt)

a) Let V and W be vector spaces and $L : V \to W$ a linear map. Recall that the rank of L is the dimension of its image $L(V) \subset W$. Show that the rank of L is the biggest number k for which $\Lambda^k L : \Lambda^k V \to \Lambda^k W$ is nonzero. (Hint: construct a convenient basis for V.)

Solution: Let $v_1, \ldots, v_n$ be a basis for V such that $\ker(L) = \langle v_{k+1}, \ldots, v_n\rangle$. The elements $Lv_1, \ldots, Lv_k$ then form a basis for $\mathrm{Im}(L)$ and k equals the rank of L. Now a basis for $\Lambda^l V$ is given by all the products $v_{i_1} \wedge \ldots \wedge v_{i_l}$ for which $1 \leq i_1 < \ldots < i_l \leq n$. Then,

$$\Lambda^l(L)(v_{i_1} \wedge \ldots \wedge v_{i_l}) = Lv_{i_1} \wedge \ldots \wedge Lv_{i_l}$$

which clearly is zero if $l > k$, for then necessarily $i_l > k$. Moreover,

$$\Lambda^l(L)(v_1 \wedge \ldots \wedge v_k) = Lv_1 \wedge \ldots \wedge Lv_k$$

is nonzero because the elements $Lv_1, \ldots, Lv_k$ are linearly independent.

b) For a nonzero vector $v \in V$ we consider for each $k \geq 0$ the linear map $v\wedge : \Lambda^k V \to \Lambda^{k+1} V$ given by $\alpha \mapsto v \wedge \alpha$. Show that its kernel is given by the image of $v\wedge : \Lambda^{k-1} V \to \Lambda^k V$. (Hint: construct a convenient basis for V.)

Solution: Since $v \neq 0$ we can complement it to a basis $v_1, \ldots, v_n \in V$, where $v_1 = v$. For $k = 0$ we know that $v\wedge : \Lambda^0 V = \mathbb{R} \to \Lambda^1 V = V$ is injective, since it maps 1 to $v \neq 0$. For $k > 0$, the elements $v_1 \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}$ lie in the kernel of $v_1\wedge$, while the elements $v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}$ with $1 < i_1 < \ldots < i_k \leq n$ are mapped by $v_1\wedge$ to a basis of $\mathrm{Im}(v_1\wedge)$. Indeed, this follows from the fact that the elements $v_1 \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}$ are all linearly independent in $\Lambda^{k+1} V$. So we see that $\ker(v_1\wedge) = \langle v_1 \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} | 1 < i_2 < \ldots < i_k \leq n\rangle$, which is precisely the image of $v_1\wedge : \Lambda^{k-1} V \to \Lambda^k V$.

Exercise 3(30 pt) Consider the two-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$.

a) Compute $\int_{S^2(r)} \omega$, where $S^2(r) := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2\}$ is the two-sphere of radius $r > 0$ in $\mathbb{R}^3$.

Solution: By Stokes we know that $\int_{S^2(r)} \omega = \int_{B(r)} dw = 3 \int_{B(r)} dx \wedge dy \wedge dz$, where $B(r) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \le r^2\}$. Using polar coordinates this gives

$$\int_{S^2(r)} \omega = 4\pi r^3.$$

b) Let $\alpha := f \cdot \omega \in \Omega^2(\mathbb{R}^3 \backslash 0)$ where f is the function given by $f(x, y, z) := (x^2 + y^2 + z^2)^{-\frac{3}{2}}$. Show that $d\alpha = 0$ and use this to conclude that $\int_{S^2(r)} \alpha$ is independent of $r \in \mathbb{R}_{>0}$. What is its value?

Solution: We have $d\alpha = df \wedge \omega + f d\omega$, and

$$\begin{aligned} df \wedge \omega =& -\frac{3}{2}(x^2 + y^2 + z^2)^{-5/2}(2xdx + 2ydy + 2zdz) \wedge (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\ =& -3f dx \wedge dy \wedge dz = -f d\omega, \end{aligned}$$

which shows that $d\alpha = 0$. Now consider the manifold $A(r_1, r_2) := \{(x, y, z) \in \mathbb{R}^3 | r_1 \le x^2 + y^2 + z^2 \le r_2\}$ for $0 < r_1 < r_2$, with boundary given by $S^2(r_2) \coprod -S^2(r_1)$. Stokes then gives us

$$0 = \int_{A(r_1,r_2)} d\alpha = \int_{S^2(r_2)} \alpha - \int_{S^2(r_1)} \alpha.$$

This shows that the integral is independent of $r \in \mathbb{R}_{>0}$. To see what it is, we take $r = 1$. There $f = 1$, hence $\alpha = \omega$, and from a) we deduce that $\int_{S^2(r)} \alpha = 4\pi$.

c) Let V be the vector field on $\mathbb{R}^3 \backslash 0$ given by $V_{(x,y,z)} := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Compute the flow φ_t^V of V and show that $(\varphi_t^V)^*\alpha = \alpha$. Use this to give another proof of the fact that $\int_{S^2(r)} \alpha$ is independent of r.

Solution: The flow of V is given by $\varphi_t^V(x, y, z) := e^t(x, y, z)$. Indeed, $\varphi_0^V = \mathrm{Id}$, and

$$\frac{d}{dt}\varphi_t^V(x, y, z) = e^t(x, y, z) = V_{e^t(x,y,z)} = V_{\varphi_t^V(x,y,z)}.$$

To show that $(\varphi_t^V)^*\alpha = \alpha$ we can do two things. We can check it directly:

$$(\varphi_t^V)^*\alpha = (\varphi_t^V)^*f \cdot (\varphi_t^V)^*\omega = e^{-3t}f \cdot e^{3t}\omega = f\omega = \alpha$$

or we can compute the Lie derivative:

$$\mathcal{L}_V\alpha = \iota_V d\alpha + d\iota_V\alpha = 0,$$

because $d\alpha = 0$ and $\iota_V\alpha = f\iota_V\omega$ and $\iota_V\omega = 0$ as one readily verifies. Since φ_t^V gives orientation preserving diffeomorphisms from $S^2(r)$ to $S^2(e^t r)$, we see that $\int_{S^2(e^t r)} \alpha = \int_{S^2(r)} (\varphi_t^V)^*\alpha = \int_{S^2(r)} \alpha$. (The fact that φ_t^V is orientation preserving follows from its explicit formula, but it is true for every flow in general because it can be continuously deformed to the identity map.)

Exercise 4(30 pt) For this exercise you may use without proof that $\int_{S^n} : H^n(S^n) \to \mathbb{R}$ is an isomorphism. Let $\pi : S^n \to \mathbb{RP}^n$ denote the quotient map and $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the antipodal map $x \mapsto -x$.

a) Show that a form $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\alpha$ for a unique $\alpha \in \Omega^k(\mathbb{RP}^n)$ if and only if $\iota^*\omega = \omega$. Deduce that $\frac{1}{2}(\omega + \iota^*\omega) \in \pi^*(\Omega^k(\mathbb{RP}^n))$ for every $\omega \in \Omega^k(S^n)$.

Solution: Since $\pi\iota = \pi$ it follows that $\iota^*\pi^* = \pi^*$. In particular, $\iota^*(\pi^*\alpha) = \pi^*\alpha$ for all $\alpha \in \Omega^k(\mathbb{RP}^n)$. Conversely, suppose that $\iota^*\omega = \omega$ for $\omega \in \Omega^k(S^n)$. The map $\pi : S^n \to \mathbb{RP}^n$ is a local diffeomorphism, and the inverse image of a point $[x] \in \mathbb{RP}^n$ consists of two points; $\pm x \in S^n$. Hence, there are neighborhoods V of $[x]$ in $\mathbb{RP}^n$ and $U_\pm$ of $\pm x$ in S^n such that $\pi|_{U_\pm} : U_\pm \to V$ is a diffeomorphism. So, there are unique $\alpha_\pm \in \Omega^k(V)$ with the property that $(\pi|_{U_\pm})^*\alpha_\pm = \omega|_{U_\pm}$. Observe that $\pi|_{U_+} = \pi|_{U_-} \circ \iota$, hence $(\pi|_{U_+})^*\alpha_- = \iota^*(\pi|_{U_-})^*\alpha_- = \iota^*\omega|_{U_-} = \omega|_{U_+} = (\pi|_{U_+})^*\alpha_+$. In particular, $\alpha_+ = \alpha_-$. We have now shown that around each point in $\mathbb{RP}^n$ there is a unique α with the desired property. By uniqueness all these locally constructed α's glue together into a globally defined $\alpha \in \Omega^k(\mathbb{RP}^n)$. The last question follows immediately since $\iota^*(\frac{1}{2}(\omega + \iota^*\omega)) = \frac{1}{2}(\iota^*\omega + \omega)$, because $\iota \circ \iota = \mathrm{Id}$.

b) If n is even and $\iota^*\omega = \omega$, show that $\int_{S^n} \omega = 0$.

Solution: Let $D^+, D^- \subset S^n$ be the upper and lower hemisphere. Then ι induces a diffeomorphism $\iota : D^+ \to D^-$, which for n even is orientation reversing. Indeed, for n even the map $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is orientation reversing, and maps an outward normal of S^n to another outward normal of S^n. Hence the restriction $\iota_{S^n} : S^n \to S^n$ is orientation reversing. Consequently:

$$\int_{S^n} \omega = \int_{D^+} \omega + \int_{D^-} \omega - = \int_{D^+} \omega + \int_{D^-} \iota^*\omega = \int_{D^+} \omega - \int_{D^+} \omega = 0.$$

c) Show that $H^n(\mathbb{RP}^n) = 0$ for all even n. Deduce that $\mathbb{RP}^n$ is not orientable for n even. (Hint: for $\omega \in \Omega^n(\mathbb{RP}^n)$ show that $\pi^*\omega$ is exact. Then use part a) to write $\pi^*\omega = d\alpha$ for some α with $\iota^*\alpha = \alpha$.)

Solution: Let $\omega \in \Omega^n(\mathbb{RP}^n)$. By a) and b) we know that $\int_{S^n} \pi^*\omega = 0$, so by the given fact about $H^n(S^n)$ we know that $\pi^*\omega = d\alpha$ for some $\alpha \in \Omega^{n-1}(S^n)$. This α need not satisfy $\iota^*\alpha = \alpha$, but we can consider $\widetilde{\alpha} := \frac{1}{2}(\alpha + \iota^*\alpha)$. We have $\iota^*\widetilde{\alpha} = \widetilde{\alpha}$, while $d\widetilde{\alpha} = \frac{1}{2}(d\alpha + \iota^* d\alpha) = \pi^*\omega$. By part a) again we can write $\widetilde{\alpha} = \pi^*\beta$ for some $\beta \in \Omega^{n-1}(\mathbb{RP}^n)$, and we have $\pi^*\omega = \pi^* d\beta$. Using a) once more, this implies $\omega = d\beta$, hence ω is exact. As ω was arbitrary, $H^n(\mathbb{RP}^n) = 0$.

Exercise 5(20 pt) Recall that a vector bundle $\pi : E \to M$ is called orientable if we can choose an orientation on each fiber, in such a way that around each point in M we can find a positively oriented frame.

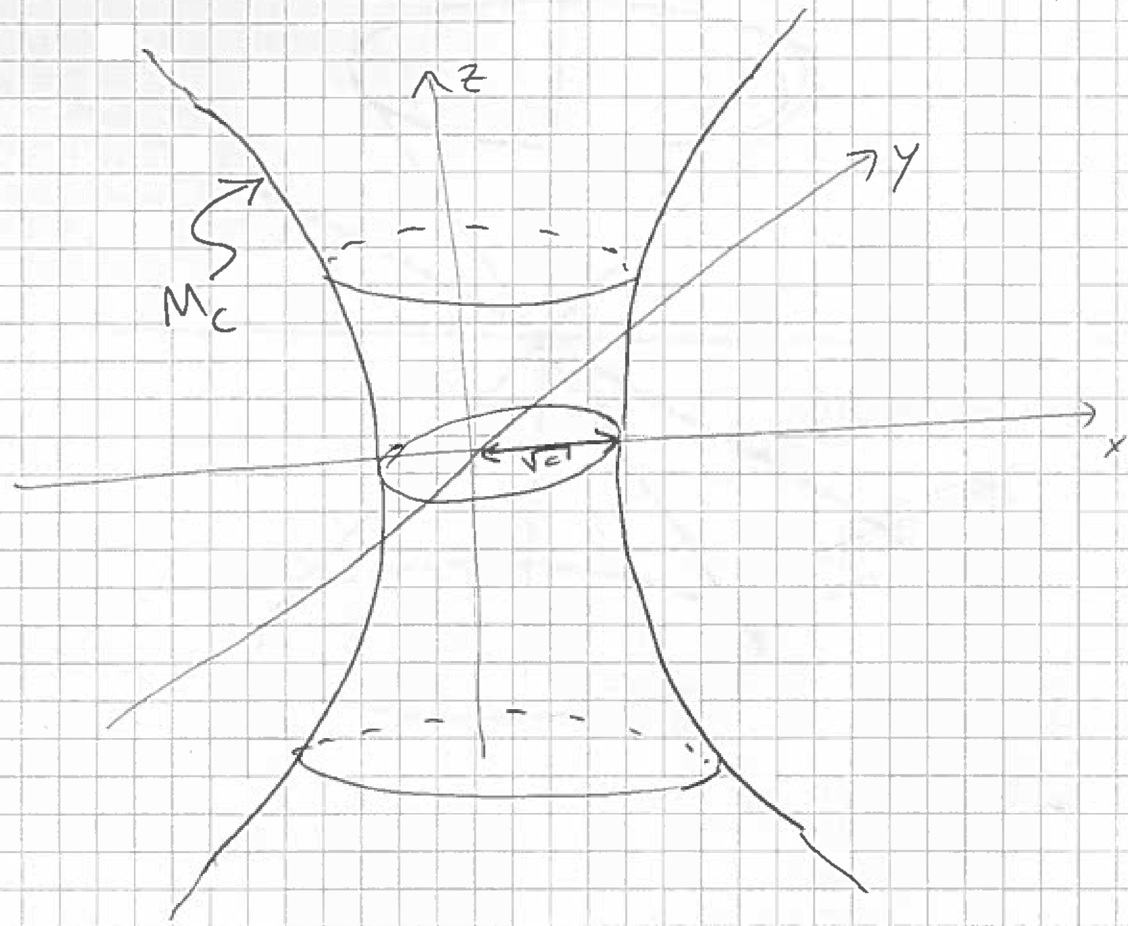
a) Show that a line bundle (i.e. a vector bundle of rank 1) is trivial if and only if it is orientable.

Solution: Clearly if E is trivial, i.e. isomorphic to $M \times \mathbb{R}$, it is orientable since we can pick a nowhere vanishing section and let that induce an orientation on each fiber. Conversely, suppose that E is orientable. Choose an open cover $\{U_\alpha\}$ of M together with positively oriented sections $s_\alpha \in \Gamma(E|_{U_\alpha})$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\rho_\alpha s_\alpha \in \Gamma(E)$, i.e. is a globally defined section of E. Let $s := \sum_\alpha \rho_\alpha s_\alpha \in \Gamma(E)$ (this is well-defined since the sum is locally finite). If $x \in M$, $s(x) = \sum_\alpha \rho_\alpha(x) s_\alpha(x)$, and all the $s_\alpha(x)$ are nonnegative in E_x with respect to the given orientation. Moreover, whenever $\rho_\alpha(x) > 0$, $s_\alpha(x) > 0$, and we know that there is at least one α for which this is true since $\sum_\alpha \rho_\alpha = 1$.

b) Show that for any line bundle E over M the line bundle $E \otimes E$ is trivial. (Hint: use a))

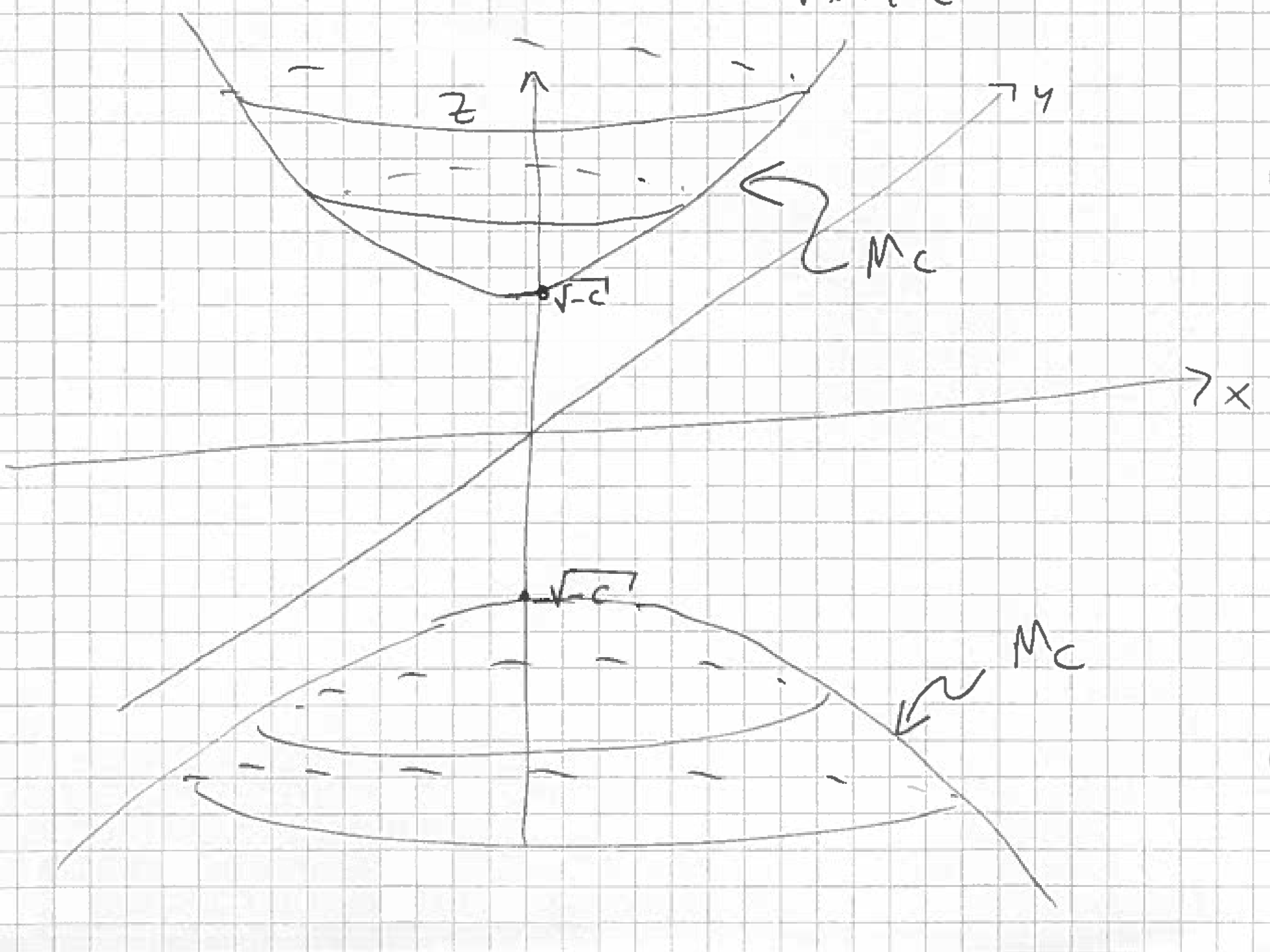
Solution: We will construct an orientation on $E \otimes E$. This is based on the following observation: if $v \in E_x$ is nonzero, it defines a nonzero element $v \otimes v \in E_x \otimes E_x$, hence an orientation on $E_x \otimes E_x$. Moreover, if $w = \lambda v$ is another nonzero element of E_x, then $w \otimes w = \lambda^2 v \otimes v$. Since $\lambda^2 > 0$ for every $\lambda \neq 0$, we see that this orientation on $E_x \otimes E_x$ is independent of the choice of nonzero vector in E_x. We endow all the fibers $E_x \otimes E_x$ of $E \otimes E$ with this orientation. This is an orientation of E, i.e. is continuous, because if e is a local frame for E, then $e \otimes e$ is a frame for $E \otimes E$ which is positive. Hence, $E \otimes E$ is oriented and so trivial by part a).

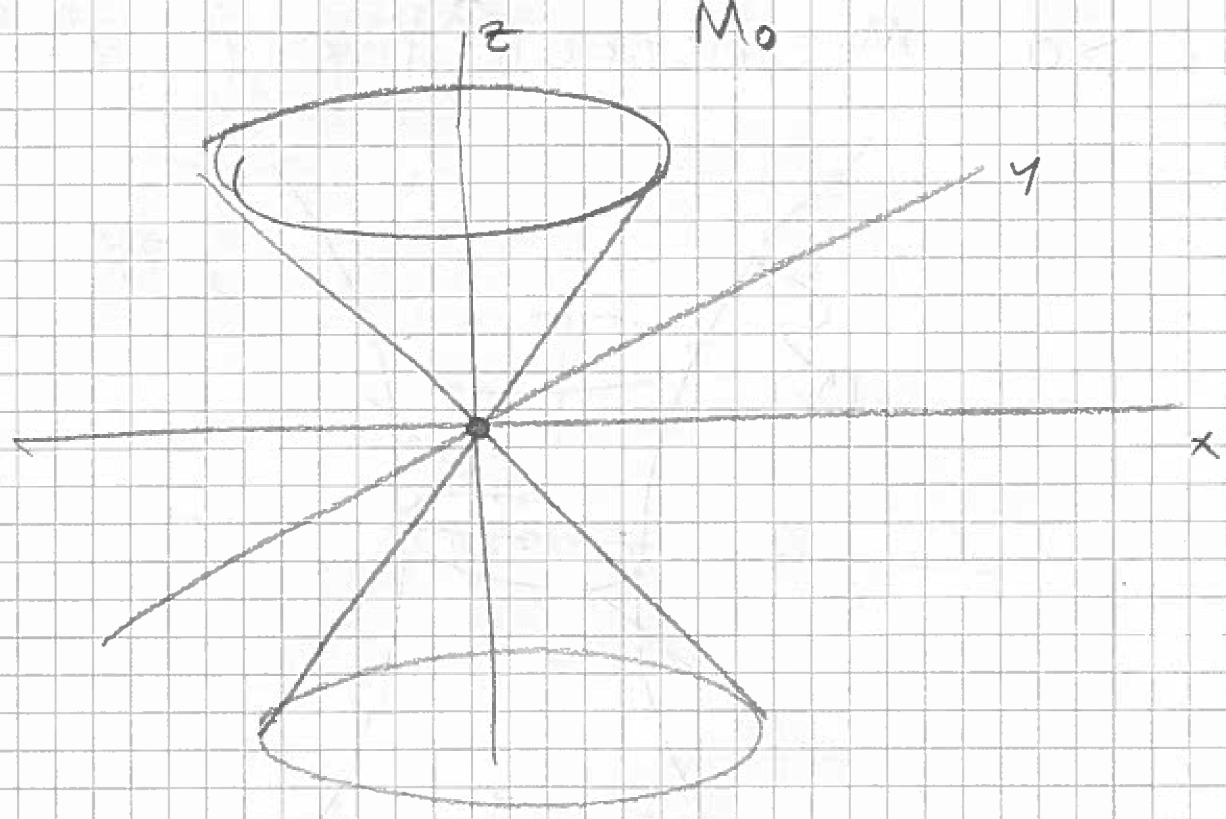
$c > 0$: $M_c = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = z^2+c\}$



$c < 0$: $M_c = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = x^2+y^2-c\}$

$\Leftrightarrow z = \pm\sqrt{x^2+y^2-c}$




z
M0
y
x