Differentiable manifolds 2016-2017: Retake

Notes:

1. **Write your name and student number** **clearly** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.
7. Every individual question is worth 10 points, giving a total of 100 points for the entire exam.

Questions

Exercise 1(20 pt) Consider the manifold $\mathbb{RP}^2$. Using homogeneous coordinates $[x_1 : x_2 : x_3]$ to denote points in $\mathbb{RP}^2$, let $U_i := \{[x_1 : x_2 : x_3] | x_i \neq 0\}$ and $\varphi_i : U_i \to \mathbb{R}^2$ given by $[x_1 : x_2 : x_3] \mapsto \frac{1}{x_i}(x_j, x_k)$, where $j < k$ are such that $\{i, j, k\} = \{1, 2, 3\}$, denote the standard atlas on $\mathbb{RP}^2$.

a) Prove that this is not an oriented atlas. Does that prove that $\mathbb{RP}^2$ is not orientable? Explain your answer.

b) Consider the polynomial $F(x_1, x_2, x_3) := x_3(x_2)^2 - x_1(x_1 - x_3)(x_1 - \lambda x_3)$ where $\lambda \in \mathbb{R}$. Give equations that describe the zero set

$$Z(F) := \{[x_1 : x_2 : x_3] \in \mathbb{RP}^2 | F(x_1, x_2, x_3) = 0\}$$

in the three charts U_1, U_2 and U_3. Determine those values of $\lambda \in \mathbb{R}$ for which $Z(F)$ is a smooth submanifold of $\mathbb{RP}^2$.

Exercise 2(20 pt) Let V be a finite dimensional vector space and $W \subset V$ a subspace. Denote by $\mathrm{Ann}(W) := \{\alpha \in V^* | \ \alpha(w) = 0 \ \forall w \in W\}$ the space of one-forms on V that annihilate W.

a) For $k \geq 0$ consider the restriction map $\Lambda^k V^* \to \Lambda^k W^*$ given by $\alpha \mapsto \alpha|_W$. Show that this is surjective and that the kernel is spanned by elements of the form $\{\alpha_1 \wedge \ldots \wedge \alpha_k | \alpha_1 \in \mathrm{Ann}(W)\}$. (Hint: construct a convenient basis for V^*.)

We will denote the kernel of the restriction map $\Lambda^k V^* \to \Lambda^k W^*$ by $I(k)$.

b) Let g be a positive definite inner product on V. Show that, using g, there is a natural way (i.e. independent of further choices) of extending elements of $\Lambda^k W^*$ to elements of $\Lambda^k V^*$. Use this do give a decomposition

$$\Lambda^k V^* = \Lambda^k W^* \oplus I(k).$$

Exercise 3(20 pt) Consider the two-form

$$\omega := \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 \backslash 0$. Recall from the Final Exam that ω is closed and that its integral over the two sphere of radius $r > 0$ is independent of r and equal to 4π. You may use these facts without proof in this exercise.

a) Let $T_1 := \{(x, y, z) | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$ denote the two-dimensional torus in $\mathbb{R}^3$ that one obtains by rotating the circle $\{(x, 0, z) \in \mathbb{R}^3 | (x - 2)^2 + z^2 = 1\}$ around the z-axis. Compute $\int_{T_1} \omega$.

b) Let $T_2 := \{(x, y, z) | (\sqrt{(x - 2)^2 + y^2} - 2)^2 + z^2 = 1\}$ denote the two-dimensional torus in $\mathbb{R}^3$ that one obtains by translating T_1 over the x-axis. Compute $\int_{T_2} \omega$.

Exercise 4(20 pt) Let $M = \mathbb{R}^2$. Consider the maps

$$\begin{aligned} H &: M \times \mathbb{R} \to M, \qquad & H(x, y, t) &:= (e^t x, y) \\ K &: M \times \mathbb{R} \to M, \qquad & K(x, y, t) &:= (x, y + tx). \end{aligned}$$

a) Show that H and K are the flows of vector fields X_H, respectively, X_K. Determine X_H and X_K explicitly.

b) Let us write $H_t : M \to M$ for the map $(x, y) \mapsto H(x, y, t)$, and similarly for K. Show that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} K_{-s} H_{-t} K_s H_t(p) = [X_H, X_K](p)$$

for every $p \in M$.

Exercise 5(20 pt) Let M be a smooth n-dimensional manifold and suppose that $X_1, \ldots, X_n$ are vector fields on M such that $X_1(p), \ldots, X_n(p)$ forms a basis of T_pM for all $p \in M$. Let $\alpha^1, \ldots, \alpha^n$ be the one-forms on M dual to the X_i, determined by the relation $\alpha^i(X_j) = \delta^i_j$.

a) Show that $[X_i, X_j] = 0$ for all i, j if and only if $d\alpha^i = 0$ for all i.

b) Suppose that $H^1_{dR}(M) = 0$. Show that the vector fields $X_1, \ldots, X_n$ are equal to the coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ for some coordinate system $(x^1, \ldots, x^n)$ on M if and only if $[X_i, X_j] = 0$.
(Hint: use the assumption together with part a) to find functions $x^1, \ldots, x^n$ on M such that $dx^i(X_j) = \delta^i_j$.)