$\boxed{\text{Exercise 1}}$:

"$\Rightarrow$" If $\gamma: I \to M$ integral curve of $X \Rightarrow \dot{\gamma}(t) = X_{\gamma(t)}$ $(\forall) t \in I \Rightarrow$

$\Rightarrow (f\circ\gamma)'(t) = (df)_{\gamma(t)}(\dot{\gamma}(t)) = (df)_{\gamma(t)}(X_{\gamma(t)}) = L_X(f)(\gamma(t)) = 0$ $\underline{(\forall) t \in I} \Rightarrow f\circ\gamma = \text{constant}$

"$\Leftarrow$" Let $p \in M$; we prove $L_X(f)(p) = 0$, i.e. $(df)_p(X_p) = 0$. Let γ_p - integral curve of X with $\gamma(0) = p$. Then $(df)_p(X_p) = \frac{d}{dt}\Big|_{t=0} \underbrace{f(\gamma_p(t))}_{\text{const}} = 0$ ☒

$\boxed{\text{Exercise 2}}$:

(a) Remark that: $\begin{cases} i_X(dy) = x, \ i_X(dx) = -y, \ i_X(dz) = 0. & (1) \\ x\,dx + y\,dy + z\,dz = 0 \text{ on } S^2. & (2) \end{cases}$

We use the main properties of i_X:

$i_X(\theta) = x(i_X(dy)\wedge dz - dy\wedge i_X(dz)) + y(i_X(dz)\wedge dx - dz\wedge i_X(dx)) + z(i_X(dx)\wedge dy - dx\wedge i_X(dy))$

$\overset{(1)}{=} x\cdot(x\,dz) + y(dz\wedge(-y)) + z\cdot(-y\,dy - x\,dx) = \underbrace{(x^2+y^2)}_{1-z^2}dz - yz\,dy - xz\,dx \overset{(2)}{=} dz$

hence $\boxed{i_X(\theta) = dz, \ d\,i_X(\theta) = 0.}$

(b) Using properties of d $\Rightarrow$ $d\theta = dx\cdot dy\wedge dz) + \text{etc} = (dx\wedge dy\wedge dz + x\,d(\overset{0}{d}y)\wedge dz + x\,dy\wedge \overset{0}{d}dz + \text{etc}$

$= dx\hat{d}y\hat{d}z + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy = 3\,dx\wedge dy\wedge dz.$

Also, $\frac{1}{3} i_X(d\theta) = i_X(dx)\wedge dy\wedge dz - dx\wedge i_X(dy)\wedge dz + dx\wedge dy\wedge i_X(dz) \underset{(1)}{=} -y\,dy\wedge dz - x\,dx\wedge dz$

$= -(y\,dy + x\,dx)\wedge dz \overset{(2)}{=} 0 \Rightarrow \boxed{d\theta = 3\,dx\wedge dy\wedge dz, \ i_X(d\theta) = 0}.$

(c) $L_X = d\,i_X(\theta) + i_X(d\theta) = 0$. Or, using the properties of L_X and (1): but $L_X(x) = -y$, $L_X(y) = x$, $L_X(z) = 0$

$L_X(\theta) = (L_X(x)\,dy\wedge dz + x\,dL_X(y)\wedge dz + x\,dy\wedge dL_X(z)) +$

$+ (L_X(y)\,dz\wedge dx + y\,dL_X(z)\wedge dx + y\,dz\wedge dL_X(x)) +$

$+ (L_X(z)\,dx\wedge dy + z\,dL_X(x)\wedge dy + z\,dx\wedge dL_X(y))$

$= -y\,dy\wedge dz + x\,\cancel{dx}\wedge dz + x\,\cancel{dz}\wedge dx + y\,\cancel{dz\wedge(-dy)} + z\,\underbrace{(-dy)\wedge dy}_{0} + z\,\underbrace{dx\wedge dx}_{0} = 0$

(d) If $\varphi^t(a,b,c) = (x(t), y(t), z(t)) \Rightarrow \begin{cases} \dot{x} = -y \\ \dot{y} = x \\ \dot{z} = 0 \Rightarrow z(t) = \text{constant} = c \end{cases} \Rightarrow \ddot{x} = -x.$

Remark that $\begin{cases} x(t) = A\cos t + B\sin t \\ y(t) = A\sin t - B\cos t \end{cases}$ does satisfy $\ddot{x} = -x$; Want $\begin{cases} x(0) = a \\ y(0) = b \end{cases} \Rightarrow \begin{cases} A = a \\ B = -b \end{cases}$

$\Rightarrow x(t) = a\cos t - b\sin(t)$, $y(t) = a\sin(t) + b\cos(t)$, $z(t) = c$ is the solution.

By uniqueness $\Rightarrow$ is $\underline{\text{the}}$ solution and $\boxed{\varphi^t(a,b,c) = (a\cos t - b\sin t, a\sin t + b\cos t, c)}$

(e) We have $\frac{d}{dt}(\varphi^t_X)^*(\theta) = (\varphi^t_X)^*(\underbrace{L_X(\theta)}_{0 \text{ by c)}}) = 0 \Rightarrow (\varphi^t)^*(\theta) = \text{cst. in } t \Rightarrow$

$\Rightarrow (\varphi^t)^*(\theta) = (\varphi^0)^*(\theta) = \theta.$

OR: Compute using properties of pull-backs: $\underline{\text{(d)}}$

$(\varphi^t)^*(\theta) = (\varphi^t)^*(x)\,d(\varphi^t)^*(y)\wedge d(\varphi^t)^*(z) + \text{etc} = \ldots = \theta.$

$= (x\cos t - b\sin t)\,d(x\sin t + y\cos t)\wedge dz$

Exercise 3: $M = f^{-1}(0)$, $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = (x^2+y^2+z^2-5)^2 + 16z^2 - 16$.

(a) Use the regular value thm. Have to check that $(df)_{x,y,z}: \mathbb{R}^3 \to \mathbb{R}$ is surjective (or, equivalently, non-zero) $\forall (x,y,z) \in M$. I.e. that.

$$\frac{\partial f}{\partial x} = 4x(x^2+y^2+z^2-5),\ \frac{\partial f}{\partial y} = 4y(x^2+y^2+z^2-5),\ \frac{\partial f}{\partial z} = 4z(x^2+y^2+z^2-5)+32z = 4z(x^2+y^2+z^2+3)$$

cannot all be 0 at some $(x,y,z) \in M$. If it is, the last eq. $\Rightarrow z = 0$. But $(x,y,z) \in M$ $\Rightarrow x^2+y^2+z^2-5 = \pm 4 \neq 0 \Rightarrow$ hence, for $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0 \Rightarrow x = y = 0$, but $(0,0,0) \notin M$ contradict!

(b) The partial derivatives at $(3,0,0)$ are:

$$\frac{\partial f}{\partial x}\Big|_{(3,0,0)} = 48,\ \frac{\partial f}{\partial y}(3,0,0) = 0,\ \frac{\partial f}{\partial z}(3,0,0) = 0$$

hence $(df)_{(3,0,0)}: \mathbb{R}^3 \to \mathbb{R}$ is $\underset{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}{\underset{\shortparallel}{(u,v,w)}} \longmapsto 48u$, hence $\mathrm{Ker}\,(df)_{(3,0,0)} = \mathrm{span}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Similarly the other one.

(c) ...

(d). Look e.g. at the lecture notes, or think:

Scratch paper: $(x^2+y^2+z^2-5)^2 + 16z^2 = 16$ holds if $\begin{cases} z = \sin(\theta) \\ x^2+y^2+z^2-5 = 4\cos\theta. \end{cases}$

$\Leftrightarrow \begin{cases} x^2+y^2 = 5+4\cos\theta - \sin^2\theta = \\ \quad = 4+4\cos\theta+\cos^2\theta \\ \quad = (2+\cos\theta)^2. \end{cases}$ ok if $x = (2+\cos\theta)\cos\alpha$, $y = (2+\cos\theta)\sin\alpha$.

Define $\phi: S^1 \times S^1 \longrightarrow M$, $\phi(e^{i\theta}, e^{i\alpha}) = ((2+\cos\theta)\cos\alpha, (2+\cos\theta)\sin\alpha, \sin\theta)$.

It is clearly (?) smooth. Check it is bijective. Use that $S^1 \times S^1 =$ compact and $M =$ Hausdorff ~~to conclude~~ (and what else?) to conclude that ϕ is a diffeom. (fill in the details!).

Exercise 4:

(a). Given f, since $\{\theta_1, \theta_2, \theta_3\}$ - basis $\Rightarrow df = g^1 \cdot \theta_1 + g^2 \cdot \theta_2 + g^3 \cdot \theta_3$ for some $g^i \in C^\infty(M)$.

Applying this to $V^i \Rightarrow df(V^i) = g^i \Rightarrow g^i = df(V^i) = L_{V^i}(f)$, hence $df = L_{V^1}(f)\,\theta_1 + $ etc.

(b) Since $\{V^1, V^2, V^3\}$ = basis, one has:

$$d\theta_1 = 2\theta_2 \wedge \theta_3 \Leftrightarrow \begin{cases} d\theta_1(V^2,V^3) = 2\theta_2\wedge\theta_3(V^2,V^3) \\ d\theta_1(V^3,V^1) = -2\theta_2\wedge\theta_3(V^3,V^1) \\ d\theta_1(V^1,V^2) = -2\theta_2\wedge\theta_3(V^1,V^2). \end{cases} \Longleftrightarrow \begin{cases} \theta_1([V^2,V^3]) = 2\theta_2\wedge\theta_3(V^2,V^3) \\ \theta_1([V^3,V^1]) = 2\theta_2\wedge\theta_3(V^3,V^1) \\ \theta_1([V^1,V^2]) = 2\theta_2\wedge\theta_3(V^1,V^2). \end{cases} \Longleftrightarrow$$

since $d\theta_1(V^i,V^j) = L_{V^i}(\theta(V^j)) - L_{V^j}(\theta(V^i)) - \theta_1([V^i,V^j]) = -\theta_1([V^i,V^j])$

with $\theta(V^j) = $ const, hence $L_{V^i}(\theta(V^j)) = 0$

$$\Longleftrightarrow \begin{cases} \theta_1([V^2,V^3]) = 2 \\ \theta_1([V^3,V^1]) = 0 \\ \theta_1([V^1,V^2]) = 0 \end{cases}$$

since $\theta_2\wedge\theta_3(V^2,V^3) = \theta_2(V^2)\theta_3(V^3) - 0 = 1$

$(\theta_2\wedge\theta_3)(V^3,V^1) = 0 - 0 = 0$.

$(\theta_2\wedge\theta_3)(V^1,V^2) = 0 - 0 = 0$.

Given the symmetry in θ_i's we obtain:

$$\begin{cases} d\theta_1 = -2\theta_2\wedge\theta_3 \\ d\theta_2 = -2\theta_3\wedge\theta_1 \\ d\theta_3 = -2\theta_1\wedge\theta_2 \end{cases} \Longleftrightarrow \begin{cases} \theta_1([V^2,V^3]) = 2 & \theta_2([V^2,V^3]) = 0 & \theta_3([V^2,V^3]) = 0 \\ \theta_1([V^3,V^1]) = 0 & \theta_2([V^3,V^1]) = 2 & \theta_3([V^3,V^1]) = 0 \\ \theta_1([V^1,V^2]) = 0 & \theta_2([V^1,V^2]) = 0 & \theta_3([V^1,V^2]) = 2 \end{cases} \Longleftrightarrow \begin{cases} [V^2,V^3] = 2V^1 \\ [V^3,V^1] = 2V^2 \\ [V^1,V^2] = 2V^3 \end{cases}$$

(recall: $X = \theta_1(X)V^1 + \theta_2(X)V^2 + \theta_3(X)V^3$ $\forall X$)

(c) Using magic Cartan: $dh_1 = d(i_V(\theta_1)) = \underbrace{L_V(\theta_1)}_{0 \text{ by hyp}} - i_V(\underbrace{d\theta_1}_{-2\theta_2\wedge\theta_3 \text{ by b)}}) = 2i_V(\theta_2\wedge\theta_3) =$
$= 2\, i_V(\theta_2)\wedge\theta_3 - 2\,\theta_2\wedge i_V(\theta_3) = 2h_2\theta_3 - 2h_3\theta_2$ and similarly the rest.

(d). We show that $R = h_1^2 + h_2^2 + h_3^2 : M \to \mathbb{R}$ has zero differential:
$\frac{1}{2}dR = h_1\,dh_1 + h_2\,dh_2 + h_3\,dh_3 \underset{c)}{\Longrightarrow} \frac{1}{4}dR = h_1(\underline{h_2\theta_3} - \underline{h_3\theta_2}) + h_2(\underline{h_3\theta_1} - \underline{h_1\theta_3}) + h_3(\underline{h_1\theta_2} - \underline{h_2\theta_1}) = 0$
$\Rightarrow dR = 0$. Since M = connected $\Rightarrow R$ = constant (why?) $\Rightarrow$ OK.

(e) By Ex 1: enough to show $L_V(h_i) = 0$ $\forall i$. I.e. $dh_i(V) = 0$. Use c). to compute $dh_1(V) = 2h_2\underbrace{\theta_3(V)}_{h_3} - 2h_3\underbrace{\theta_2(V)}_{h_2} = 0$ and similarly (or by symmetry) the rest.

(f) Compute $(dh)_p(V_p^1)$ using again c) ~~...~~
and $(dh)_p(V_p^1) = ((dh_1)_p(V_p^1), (dh_2)_p(V_p^1), (dh_3)_p(V_p^1)) = dh_1(V_p^1)\frac{\partial}{\partial x} + dh_2(V_p^1)\frac{\partial}{\partial y} + dh_3(V_p^1)\frac{\partial}{\partial z}$

where $\begin{cases} dh_1(V_p^1) = 2h_2\theta_3(V_p^1) - 2h_3\theta_2(V_p^1) = 0 \\ dh_2(V_p^1) = 2h_3\cdot 1 - 0 = 2h_3 \\ dh_3(V_p^1) = 0 - 2h_2\cdot 1 = -2h_2 . \end{cases}$ $\Rightarrow (dh)_p(V_p) = 2h_3(p)\left(\frac{\partial}{\partial y}\right)_{h(p)} - 2h_2(p)\left(\frac{\partial}{\partial z}\right)_{h(p)} = 2E^1_{h(p)}$.

(g) : By (f), since $\{E^1, E^2, E^3\}$ - span TS^2 at each point $\Rightarrow h$ is a submersion. Using the local form of submersions $\Rightarrow \mathrm{Im}(f)$ is open in S^2.
But M = compact $\Rightarrow \mathrm{Im}(f)$ = compact $\Rightarrow \mathrm{Im}(f)$ - closed in S^2.
$\Rightarrow \mathrm{Im}(f) = S^2$ (since S^2 = connected!) hence f = surjective.
For a fiber $f^{-1}(q)$ (non-empty because f = surjective!), choosing $p \in h^{-1}(q)$, we consider $\gamma = \gamma_p$ = the integral curve of V through p.
~~Since~~ Because of e), $\Rightarrow \gamma : \mathbb{R} \to \underbrace{h^{-1}(q)}_{\text{1-dim. compact connected mfd!}}$.

There are several ways to continue from here. E.g. show that $\exists T$ s.t. $\gamma(T) = p$, then remark that $\gamma(t) = \gamma(t+T)$ $\forall t$. Choosing T minimal $\Rightarrow$
$\Rightarrow$ get a map $S^1 \to h^{-1}(q)$, $e^{i\theta} \longmapsto \gamma(\frac{T}{2\pi}t)$. Immersion, bijection from a compact to Hausdorff $\Rightarrow$ diffeomorphism.

(h) Several possibilities. E.g. take another Lie group with the same Lie algebra: $SO(3)$. Equivalently, $M' = S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$.