

Exam Manifolds-retake; January 3rd, 2018

Note: The questions in blue are worth 1 point, and the rest 0.5 points. The points below add up together to 13.5, so you do not have to solve everything to get the maximum mark of 10! Also, please be aware: blue really means that "it is worth more points", and not that "it is more difficult".

Exercise 1. Compute the flow of $X = -x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$.

Exercise 2. Show that $M := \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 4\}$ is an embedded submanifold of \mathbb{R}^3 and compute the tangent space $T_p M \subset \mathbb{R}^3$ at the point $p = (1, 1, -1)$.

Exercise 3. Assume that M is an S^1 -manifold, i.e. a manifold together with an action of the circle S^1 - i.e. a smooth map

$$M \times S^1 \rightarrow M, \quad (\lambda, p) \mapsto \lambda \cdot p$$

such that $p \cdot 1 = p$ and $(p \cdot \lambda) \cdot \lambda' = p \cdot (\lambda \lambda')$ for all $\lambda, \lambda' \in S^1, p \in M$. We define the following vector field on M :

$$V_p = \left. \frac{d}{dt} \right|_{t=0} p \cdot e^{2\pi i t}. \quad (0.1)$$

- Write down the flow of V using the action of S^1 on M .
- Deduce that a point $p \in M$ is a fixed point of the action, i.e. satisfies $p \cdot \lambda = p$ for all $\lambda \in S^1$, if and only if p is a zero of V , i.e. satisfies $V_p = 0$.
- Assume now that the action is free i.e. that it has no fixed points (or, cf. the previous point, V has no zeroes). Show that for each $p \in M$, the orbit through p :

$$\mathcal{O}_p := \{p \cdot \lambda : \lambda \in S^1\} \quad (0.2)$$

is an embedded submanifold of M diffeomorphic to S^1 .

Exercise 4. Consider again an S^1 -manifold M . A differential form $\omega \in \Omega^\bullet(M)$ is called:

- equivariant: if $m_\lambda^*(\omega) = \omega$ for all $\lambda \in S^1$, where $m_\lambda : M \rightarrow M$ is the map $p \mapsto p \cdot \lambda$.
- horizontal: if $i_V(\omega) = 0$.
- basic form: if it is both horizontal as well as equivariant.

Show that:

- if ω is equivariant, then so is $d\omega$.
- ω is equivariant if and only if $L_V(\omega) = 0$.

- (c) if ω is basic, then so is $d\omega$.
- (d) in general, it is not true that if ω is horizontal than so is $d\omega$ (provide a counter-example, e.g. using the last homework).

Exercise 5. Assume now that M is a principal S^1 -bundle, i.e. an S^1 -manifold with the property that the action is free and that the orbits (0.2) of the action are precisely the fibers of a submersion

$$h : M \rightarrow B,$$

with values in some other manifold B (usually called the base of the bundle). Show that

- (a) The pull-back map $h^* : \Omega^\bullet(B) \rightarrow \Omega^\bullet(M)$ is injective.
- (b) For any differential form $\eta \in \Omega^k(B)$, its pull-back $\omega := h^*(\eta)$ is a basic form on M .
- (c) h^* is a linear isomorphism between $\Omega^\bullet(B)$ and the space of basic forms on M .

Exercise 6. Assume again that M is a principal S^1 -bundle, with corresponding submersion $h : M \rightarrow B$, and corresponding vector field (0.1).

We choose a 1-form $\omega \in \Omega^1(M)$ which is equivariant and satisfies $\omega(V) = 1$. Show that

- (a) $d\omega \in \Omega^2(M)$ is a basic 2-form.
- (b) denoting by $\eta \in \Omega^2(B)$ the 2-form on characterized by

$$d\omega = h^*(\eta)$$

(which exists and is unique by the previous exercise), η is a closed 2-form on B .

- (c) while η is built using any ω which was equivariant and satisfied $\omega(V) = 1$, prove that the cohomology class $[\eta] \in H^2(B)$ does not depend on the choice of ω .

(this cohomology class is known as "the first Chern class" of the principal S^1 -bundle).

Exercise 7. Returning now to the Hopf fibration and the last homework(s),

$$h : S^3 \rightarrow S^2,$$

we are in the case of a principal S^1 -bundle with the action given by $(z_0, z_1) \cdot \lambda = (\lambda z_0, \lambda z_1)$, and with corresponding vector field (as in the last homework):

$$V = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - t \frac{\partial}{\partial z} + z \frac{\partial}{\partial t}.$$

Show that

- (a) $\omega = -y \cdot dx + x \cdot dy - t \cdot dz + z \cdot dt \in \Omega^1(S^3)$ is equivariant and satisfies $\omega(V) = 1$.
- (b) compute the resulting form $\eta \in \Omega^2(S^2)$ (cf. the previous exercise).
- (c) show that $[\eta] \in H^2(S^2)$ is non-zero and try to draw conclusions on the Hopf fibration (e.g.: can it be isomorphic to the product fibration $S^1 \times S^2$?)