EXAM DIFFERENTIAL MANIFOLDS, JANUARY 29 2007, 9:00-12:00

READ THIS FIRST

- Put your name and student number on every sheet you hand in.
- You may do this exam either in English or in Dutch. Your grade will not only depend on the correctness of your answers, but also on your presentation; for this reason you are strongly advised to do the exam in your mother tongue if that possibility is open to you.
- Be clear and concise (and so avoid irrelevant discussions).
- Do not forget to turn this page: there are also problems on the other side.
- I will soon post a set of worked solutions (perhaps later today) on `http://www.math.uu.nl/people/looijeng/smoothman06.html`

(1) Let $f : M \to N$ be a C^∞-map between manifolds. Prove that $F : M \to M \times N$, $F(p) = (p, f(p))$ is an embedding.

(2) Let $U \subset \mathbb{R}^m$ be open and let $f : U \to \mathbb{R}$ be a C^∞-function with the property that $df(p) \neq 0$ for every $p \in U$ with $f(p) = 0$, so that (by the implicit function theorem) $f^{-1}(0)$ is a submanifold.
  (a) Prove that this submanifold is orientable.
  (b) Give an example of a surface in $\mathbb{R}^3$ that is not orientable (and conclude that it cannot arise in the above manner).

(3) Let $f : N \to M$ be a C^∞-map between manifolds with N oriented compact and of dimension n and let α be an n-form on M. Prove that if $H : \mathbb{R} \times M \to M$ is a flow, then $\int_N f^* H_t^* \alpha$ is constant in t. (Hint for at least one way to do this: consider the pull-back of α under the map $\mathbb{R} \times N \to M$, $(t, p) \mapsto H_t f(p)$.)

(4) Let $f : M \to N$ be a C^∞-map between manifolds and let V be a vector field on N. A *lift* of V over f is a vector field $\tilde{V}$ on M with the property that $D_p f(\tilde{V}_p) = V_{f(p)}$ for all $p \in M$.

(a) Prove that f is a submersion at p, then there is an open neighborhood $U \ni p$ in M such that V has a lift over $f|_U : U \to N$.

(b) Prove that if $U \subset M$ is open and $\tilde{V}_0, \dots, \tilde{V}_k$ are lifts of V over $f|_U$, then any convex linear combination of these is also one, that is, if $\phi_0, \dots, \phi_k : U \to \mathbb{R}$ are C^∞-functions with $\sum_i \phi_i$ constant 1, then $\sum_i \phi_i \tilde{V}_i$ is also a lift of V.

In the remaining parts of this problem we assume that M and N are compact and that f is a submersion. Since N is compact, V generates a flow $H : \mathbb{R} \times N \to N$.

(c) Prove that there exists a lift $\tilde{V}$ of V over f.

(d) Let $\tilde{H} : \mathbb{R} \times M \to M$ be the flow generated by this lift $\tilde{V}$. Prove that $f\tilde{H}_t = H_t f$.

(5) Let M be a m-manifold and μ a nowhere zero m-form on M. Prove that M has an atlas such that every chart (U, κ) in that atlas has the property that $\mu|_U = \kappa^*(dx^1 \wedge \cdots \wedge dx^m)$. Prove that any coordinate change of this atlas (a diffeomorphism from an open subset of $\mathbb{R}^m$ to another) has Jacobian a matrix of determinant constant 1.