Exam Differential manifolds Jan 2007: Solutions

1. Let $f : M \to N$ be a C^∞-map between manifolds. Prove that $F : M \to M \times N$, $F(p) = (p, f(p))$ is an embedding.
According to a theorem it suffices to show that F is an injective immersion which maps M homeomorphically onto the subspace $F(M)$ of N. To this end we consider the projection $\pi : M \times N \to M$. Then πF is the identity. So F is injective and an immersion (for $D_{F(p)} D\pi_{F(p)} : T_pM \to T_pM$ is the identity map). The maps π and F are continuous and hence so are their restrictions $M \to F(M)$ and $F(M) \to M$. So F maps M homeomorphically onto $F(M)$.

2. Let $U \subset \mathbb{R}^m$ be open and let $f : U \to \mathbb{R}^m$ be a C^∞-function with the property that $df(p) \neq 0$ for every $p \in U$ with $f(p) = 0$, so that (by the implicit function theorem) $f^{-1}(0)$ is a submanifold.
2a. Prove that this submanifold is orientable.
Let $p \in f^{-1}(0)$. Then the tangent space of $f^{-1}(0)$ at p is the hyperplane in $T_p\mathbb{R}^m$ that is the kernel of the linear form $df_p : T_p\mathbb{R}^m \to \mathbb{R}$. The orientation of $\mathbb{R}^m$ and the observation that this hyperplane is the boundary of the halfspace defined by $df_p \le 0$ give an orientation of the kernel of df_p. We thus obtain in a canonical fashion an orientation of the tangent bundle of $f^{-1}(0)$.
2b. Give an example of a surface in $\mathbb{R}^3$ that is not orientable (and conclude that it cannot arise in the above manner).
The Möbius band is not orientable, but can be embedded in $\mathbb{R}^3$.

3. Let $f : N \to M$ be a C^∞-map between manifolds with N oriented compact and of dimension n and let α be an n-form on M. Prove that if $H : \mathbb{R} \times M \to M$ is a flow, then $\int_N f^* H_t^* \alpha$ is constant in t. (Hint: consider the pull-back of α under the map $F : \mathbb{R} \times N \to M$, $(t,p) \mapsto H_t f(p)$.)
Here are two proofs, the first one uses the hint. In that case, consider the m-form $F^*\alpha$. Since α is closed, so is $F^*\alpha$. We know that then exists an n-form β on N such that $F^*\alpha - \pi_N^*\beta$ is exact, i.e., of the form $d\gamma$, where $\pi_N : \mathbb{R} \times N \to N$ is the projection. Now is

$$\int_N f^* H_t^* \alpha = \int_{\{t\}\times N} F^*\alpha = \int_{\{t\}\times N} \pi_N^*\beta + \int_{\{t\}\times N} d\gamma = \int_N \beta$$

(for $\int_{\{t\}\times N} d\gamma = 0$ by Stokes) and hence constant. Another proof uses the Lie derivative: if V is the infinitesimal generator of the flow, then

$$\frac{d}{dt}\int_N f^* H_t^* \alpha = \frac{d}{ds}\Big|_{s=0} \int_N f^* H_t^* H_s^* \alpha = \int_N f^* H_t^* \frac{d}{ds}\Big|_{s=0} H_s^* \alpha =$$
$$= \int_N f^* H_t^* \mathcal{L}_V \alpha = \int_N f^* H_t^* d\iota_V \alpha = \int_N d(f^* H_t^* \iota_V \alpha) = 0,$$

again by Stokes' theorem, and so $\int_N f^* H_t^* \alpha$ is constant in t.

4. Let $f : M \to N$ be a C^∞-map between manifolds and let V be a vector field on N. A lift of V over f is a vector field $\tilde{V}$ on M with the property that $D_p f(\tilde{V}_p) = V_{f(p)}$ for all $p \in M$.

4a. Prove that f is a submersion at p, then there is an open neighborhood $U \ni p$ in M such that V has a lift over $f|_U : U \to N$.

If f is a submersion at p, then we can find charts κ at p and λ at $f(p)$ so that $\lambda f \kappa^{-1}$ is the restriction of the projection $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$. If V is at $f(p)$ given by $\sum_{i=1}^n V^i \partial/\partial\lambda^i$, then a lift $\tilde{V}$ of V at p is given by $\sum_{i=1}^n (V^i f)\partial/\partial\kappa^i$.

4b. Prove that if $U \subset M$ is open and $\tilde{V}_0, \dots, \tilde{V}_k$ are lifts of V over $f|_U$, then any convex linear combination of these is also one, that is, if $\phi_0, \dots, \phi_k : U \to \mathbb{R}$ are C^∞-functions with $\sum_i \phi_i$ constant 1, then $\sum_i \phi_i \tilde{V}_i$ is also a lift of V.

We have $D_p f(\tilde{V}_{i,p}) = V_{i,f(p)}$, $i = 0, \dots, k$, and so

$$D_p f(\sum_{i=0}^{k} \phi(p)\tilde{V}_{i,p}) = \sum_{i=0}^{k} \phi(p) D_p f(\tilde{V}_{i,p}) = \sum_{i=0}^{k} \phi(p) V_{i,f(p)} = V_{i,f(p)}.$$

In the remaining parts of this problem we assume that M and N are compact and that f is a submersion. Since N is compact, V generates a flow $H : \mathbb{R} \times N \to N$.

4c. Prove that there exists a lift $\tilde{V}$ of V over f.

According to 4a) we can cover M by open subsets U_α such that V has a lift $\tilde{V}_\alpha$ over $f|_{U_\alpha}$. Since M is compact, M is covered by finitely many such open subsets $U_{\alpha_0}, \dots, U_{\alpha_k}$. If $\{\phi_i\}_{i=0}^k$ is a partition of 1 subordinate to this covering, then 4b) implies that $\tilde{V} := \sum_{i=1}^k \phi_i \tilde{V}_{\alpha_i}$ is a lift of V over f.

4d. Let $\tilde{H} : \mathbb{R} \times M \to M$ be the flow generated by this lift $\tilde{V}$. Prove that $f\tilde{H}_t = H_t f$.

Fix $p \in M$. If $\tilde{\gamma}_p(t) := \tilde{H}_t(p)$, then we have

$$(f\tilde{\gamma})\dot{}_p(t) = Df(\dot{\tilde{\gamma}}_p(t)) = Df(\tilde{V}_{\tilde{\gamma}_p(t)}) = V_{f\tilde{\gamma}_p(t)}$$

and so $f\tilde{\gamma}_p$ is the integral curve γ_p of V through $f(p)$. In other words, $f\tilde{H}_t(p) = H_t f(p)$. Since this is true for every $p \in M$, it follows that $f\tilde{H}_t = H_t f$.

5. Let M be a m-manifold and μ a nowhere zero m-form on M. Prove that M has an atlas such that every chart (U, κ) in that atlas has the property that $\mu|_U = \kappa^*(dx^1 \wedge \cdots \wedge dx^m)$. Prove that any coordinate change of this atlas (a diffeomorphism from an open subset of $\mathbb{R}^m$ to another) has Jacobian a matrix of determinant constant 1.

If κ is any chart of M at p, then $\kappa_*\mu$ takes the form $\phi(x)dx^1 \wedge \cdots dx^m$for some nowhere zero fuction ϕ on the range of κ. If $\tilde{x}_1$ is a fuction (on an open subset of $\mathbb{P}^m$ such that $\frac{\partial \tilde{x}_1}{\partial \tilde{x}_1} = \phi$, then μ is at p equal to $d\tilde{x}^1 \wedge dx^2 \wedge \cdots \wedge dx^m$.

The Jacobian matrix of the map $(x^1, \ldots, x^m) \mapsto (\tilde{x}^1, x^2, \ldots, x^m)$ is $\frac{\partial \tilde{x}_1}{\partial \tilde{x}_1} = \phi$ and hence nonzero. So by the inverse function therorem, $(\tilde{x}^1, x^2, \ldots, x^m)$ is also a chart of M at p.

Prove that any coordinate change of this atlas (a diffeomorphism from an open subset of $\mathbb{R}^m$ to another) has Jacobian a matrix of determinant constant 1.

Any coordinate change is a diffeomorphism h from an open subset $U \subset \mathbb{R}^m$ onto another open subset $U' \subset \mathbb{R}^m$ that has the property that $h^* dx^1 \wedge \cdots \wedge dx^m = dx^1 \wedge \cdots \wedge dx^m$. This means that the Jacobian matrix of h has determinant 1.